# Modeling Overlapping Generations in a Complementarity Format*

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**Abstract**

This paper provides an introduction to numerical simulation of overlapping generations models with perfect foresight and finite lifetimes. The paper illustrates the advantages of the complementarity framework for this class of problems. Three GAMS programs demonstrate how these models can be formulated in applied research. Issues addressed in the paper include calibration to base year data, representation of international trade, bequests, government tax and expenditure policies, and labor/leisure choice with endogenous retirement.

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# 1 Introduction

Overlapping generations (OLG) models with consistent expectations and finite lifetimes have become a standard tool for applied policy analysis since the seminal contribution of Auerbach and Kotlikoff [1987]. A number of researchers have subsequently adopted the AK-OLG framework for applied work (Altig and Carlstrom [1999], Broer et al. [1994], Altig et al. [2001], Keuschnigg and Kohler [1994], Knudsen et al. [2000], Kotlikoff et al. [1999], Kotlikoff et al. [2001], Lau [2000], Miles [1999], Wendner [2001]).[1] The original contribution and many of the subsequent papers in this field were based on model-specific solution algorithms. The purpose of our paper is to demonstrate the usefulness of a general-purpose complementarity format for approximating infinite horizon equilibria for overlapping generations models. Our objective is pedagogic – the essential equations for a few models are presented in a compact and accessible format along with computer programs which concretely illustrate the models. We believe that this approach is of interest to applied economists due to the availability of "off the shelf" software for processing models in the complementarity format (see Brooke et al. [1998], Rutherford [1995], Rutherford [1999], Ferris and Munson [2000]).[2]

Our paper begins with a simple OLG-exchange model of the type investigated by Kehoe and Levine [1985] to illustrate the basic issues in OLG modeling. We show how to maintain simplicity by imposing steady-state conditions as terminal constraints on the model rather than solving separately for the terminal steady-state equilibrium. We also show how to incorporate multiple household types and how to represent a baseline steady-state growth path with exogenous time profiles for consumption, income and bequests.

The second section in this paper deals with extending the model to incorporate production, labor-leisure choice and endogenous retirement decisions. This section is oriented toward models calibrated to actual data rather than stylized parameters which give "reasonable" results. The aim of this approach is that using our analytical framework, an OLG model can be implemented from standard social accounting data and it need not involve substantially more work than building a representative agent (Ramsey) model for policy analysis. We illustrate our approach with a sample application of the model to represent intergenerational issues arising in the analysis of fundamental tax reform.

There are several novelties in this paper. It is the first paper to guide applied researchers as how to exploit the complementarity framework for solving OLG models. The

[1] We focus on intertemporal models with consistent expectations and finite life-spans. Our exposition therefore does not address the Blanchard [1985] framework which is based on an assumed constant probability of death. Nor does our paper consider the Fullerton and Rogers [1993] framework which is based on imperfect foresight. We also abstract from uncertainty as has been introduced into the OLG setting by, among other, Hubbard et al. [1994].

[2] Computational issues involved in calibrating and solving OLG models have previously been addressed by Doquier and Liégeois [2001] and Wendner [1999] . Neither of these papers, however, relate to the complementarity framework. See Kotlikoff [2000] for a historical perspective on algorithms and applications.

key insight in this approach is that by using a solution method which treats the transition problem as a simultaneous system of inequalities we can exploit the enormous advances in numerical methods for these problems which have been made in recent years (Ferris and Munson [2000], Ferris et al., eds [2001]). Another contribution of this paper is that it is the first "how to" introduction to overlapping generations modeling. The paper begins with the essential elements of the model and follows a set of simple extensions which should appeal to researchers who are interested in the insights afforded by OLG methods but are unable to devote the time required to develop model-specific solution algorithms.

The structure of the remainder of the paper is as follows: Section 2 deals with the pure exchange overlapping generations model, including extensions for heterogeneous households and bequests. Section 3 deals with the production model, including labor-leisure choice, endogenous retirement and the calibration of such models to a base year data set. Section 4 provides a worked example of how to use an OLG model to model fundamental tax reform. Section 5 concludes. Appendices provide GAMS programs for each of the models presented in the text.[3]

## 2 The pure exchange model

We start out by presenting a simple setting in which agents engage in exchange, but there is no domestic capital market or production. A household of generation $g$ is born at the beginning of year $t = g$, lives for $N + 1$ years, and is endowed with an amount $\omega_{g,t}$ of the consumption good in each period $g \leq t \leq g + N$. Households maximize the utility of lifetime consumption subject to the present value of their endowments:

$$\begin{aligned} \max_{c_{g,t}} u_g\left(c_{g,t}\right) &= \sum_{t=g}^{g+N} \left(\frac{1}{1+\rho}\right)^{t-g} \frac{c_{g,t}^{1-\theta}}{1-\theta} \\ s.t. \qquad \sum_{t=g}^{g+N} p_t c_{g,t} &\leq \sum_{t=g}^{g+N} p_t \omega_{g,t}, \end{aligned} \tag{1}$$

where $c_{g,t}$ is consumption, $\rho$ is the utility discount rate, $1/\theta$ is the intertemporal elasticity of substitution, and $p_t$ is the present value price index.

We proceed by showing how the model may be calibrated to a given level of aggregate consumption and used to calculate the effects of a change in one of the exogenous parameters. Subsequently, we show how the model may be extended to include bequests and multiple households in each generation.

[3] Various versions of the models presented here can be found at the web site `http://nash.colorado.edu/tomruth/olgmcp`. The MPSGE programs generate figures and include various options discussed in the text but not included in the appendices.

## 2.1 Matching aggregate data

There is an indirect relationship between the households' maximization problem and the aggregate economy. For simplicity, we assume that the economy is in an initial steady-state equilibrium where endowments of each successive generation grow at the constant rate $\gamma$ , so that $\omega_{g+1,t+1} = (1+\gamma)\,\omega_{g,t}$. To illustrate the effects of introducing permanent imbalances of the kind that arise when calibrating a model to a steady state with a surplus or deficit in the trade balance or government budget, it is useful to consider the case where there is permanent trade deficit, $B_t$ which, to be consistent with the steady-state assumption grows at rate $\gamma$, so $B_{t+1} = (1+\gamma)\,B_t$.

A steady state can be fully characterized by the behavior of a single generation, so to calibrate the baseline we limit our attention to generation zero and years $t = 0, 1, ..., N$. Consequently, we omit subscript $g$ when referring to calibrated values of household variables. The baseline equilibrium must satisfy the following system of equations:

$$\frac{\lambda}{(1+\bar{r})^t} = \frac{c_t^{-\theta}}{(1+\rho)^t} \tag{2}$$

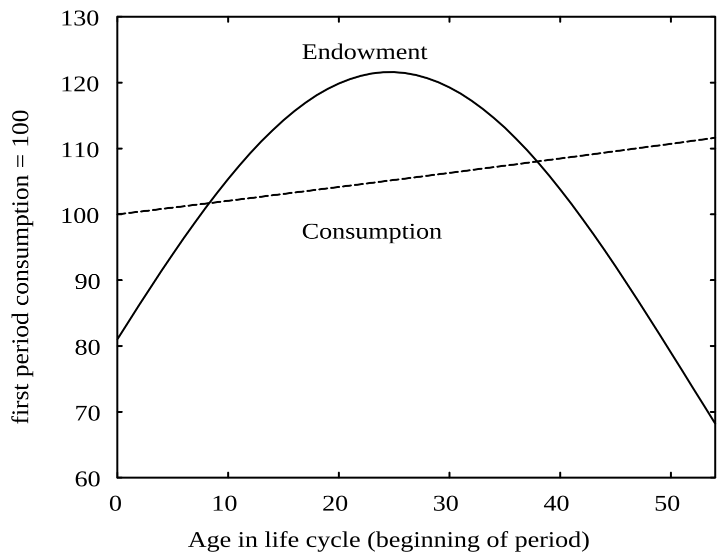
$$\sum_{t=0}^{N} \frac{c_t}{(1+\bar{r})^t} = \sum_{t=0}^{N} \frac{\omega_t}{(1+\bar{r})^t} \tag{3}$$

$$\sum_{t'=0}^{N} \frac{c_{t'}}{(1+\gamma)^{t'}} = \sum_{t'=0}^{N} \frac{\omega_{t'}}{(1+\gamma)^{t'}} + B, \tag{4}$$

where $\lambda$ is the marginal utility of income, and $\bar{r}$ is the steady-state interest rate. Equations (2) and (3) are implied by (1) and state that consumption grows at a constant rate over the life cycle and that the present value of consumption equals the present value of income. From the consumption and endowment profiles of the representative generation we can use the steady-state relationship to recover the aggregate values in year zero. Equation (4) states that aggregate demand equals aggregate supply plus the trade deficit. The summation index in this equation, $t'$, refers to generations of age $t'$ in year zero so $c_{t'}$ is the consumption level of the reference generation at age $t'$. The common term entering in the denominators on both sides of (4) accounts for steady-state growth in the size of successive generations: $1/(1+\gamma)^{t'}$ is the relative size of the generation which entered the economy $t'$ years prior to year zero.

In order to calibrate the model to an exogenous steady-state growth path, we pose the problem with $\lambda$, $c_t$, and $\rho$ as endogenous variables and solve it as a nonlinear system of equations. Figure 1 shows the calibrated consumption profile for a given endowment profile, a 55-year life-span, an interest rate, $\bar{r}$, of 5%, an aggregate growth rate, $\gamma$, of 1%, and a trade deficit equal to 1% of aggregate endowments. The consumption and endowment profiles are then extrapolated to set up the complete baseline reference path encompassing all the overlapping generations. Solving the complete model without any parameter changes reproduces the baseline equilibrium, while an exogenous shock to the

Figure 1: Endowment and calibrated consumption profile




level of endowments, the utility discount rate, or the world market interest rate produces a transition path to a new steady state.

## 2.2 Closing the model

A numerical model in discrete time can only be solved for a finite number of periods. To implement the full overlapping generations framework and calculate the transition to a new steady state it is therefore necessary to describe the special characteristics of the generations alive in the first and last periods. This implies deriving asset positions along the initial steady-state baseline and imposing terminal conditions to approximate the infinite horizon. It is also necessary to specify how capital markets relate to the trade imbalance. Here we assume that domestic and foreign goods are perfect substitutes and that there is an international bond market that finances the trade deficit. We further assume that this market is perfect in the sense that the interest rate is fixed at $\bar{r}$ so that the future value prices of imports and exports are constant.[4]

Let $a_t$ denote the present value of asset holdings over the life cycle of generation zero for $t = 0, 1, ...N$. We can compute this number by summing up the present value of

[4]Simulating a closed economy amounts to setting $B = 0$ and omitting the possibility of international trade. Alternatively, one could model period by period balance of payments constraints by representing $B$ as an exogenous endowment and letting the current value price of foreign exchange vary over time.

income less consumption for years $0, 1, ..., t-1$:

$$a_t = \sum_{t'=0}^{t-1} \frac{\omega_{t'} - c_{t'}}{(1+\bar{r})^{t'}}. \tag{5}$$

The value of assets held in year zero by a generation of age $t$ is then given by

$$m_t = a_t \left( \frac{1+\bar{r}}{1+\gamma} \right)^t, \tag{6}$$

where $a_t$ is the asset holding of generation 0 at age $t$. Division by $(1+\gamma)^t$ in this last equation accounts for the relative size of a generation born $t$ years prior to year zero, and multiplication by $(1+\bar{r})^t$ discounts asset value from year $-t$ to year zero. There is no domestic capital in the exchange model so the total value of domestic assets sums to zero at any point in time. The base year trade imbalance implies that households have a net holding of foreign assets. An important implication is that the total value of assets held in year zero, $\bar{A}$, is equal to the present value of the future trade deficits over the infinite horizon (see Appendix A for a proof):

$$\bar{A} = \sum_{t=0}^{N} m_t = \frac{1+\bar{r}}{\bar{r}-\gamma} B. \tag{7}$$

The intuition for this equation is that in order to finance the trade deficit the value of assets in the economy must correspond to the outside claim implied by the deficit. In other words, for the trade deficit to be consistent with a steady-state equilibrium, the economy must have in the past run a trade surplus thereby building up the foreign assets that finance the future deficit.

The values of $m_t$ describe the asset positions of generations alive in the first model period. These assets are modeled as exogenous endowments of period 0 goods. Similarly, the values of $m_t$ are extrapolated forward to describe the asset positions in the terminal model period (year $T$). A shock to the model may change the demand and supply for savings at a given interest rate and consequently the profile of asset holdings and the trade deficit in the new steady state. For this reason, terminal assets are modeled as endogenous variables chosen to ensure that the model achieves a steady-state growth path in the final period.[5] We choose a sufficiently large value for $T$ such that the model very nearly reaches a new steady state by the end of year $T-N$.[6] This implies that the percentage change

[5]Domestic and foreign goods are perfect substitutes, so their relative prices are fixed, and under an assumption of a fixed international interest rate, present value prices decline at a constant rate. A constant international interest rate further implies that new generations adjust immediately to a new steady state following a permanent shock to the economy at time zero. Transition effects relate only to generations who entered the economy prior to year zero.

[6]Throughout the paper models are solved for 150 years which is found sufficient to settle on a new

in welfare, as measured by the equivalent variation, $ev_{\hat{g}}$, of each of the generations living beyond the terminal period are of equal magnitude:

$$ev_{\hat{g}} = ev_{\hat{g}-1} \qquad \text{for } T - N < \hat{g} \leq T, \tag{8}$$

and the value of assets at the end of period $T$, $m_{T,\hat{g}}$, is modified accordingly. In this model where domestic and foreign goods are perfect substitutes, an intertemporal balanced budget is assured as a result of these termination conditions. Generation $T-N$ is subject to a budget constraint within the model horizon, and if the households living beyond the model horizon have the same welfare level, they must also be on the same balanced budget because their per-capita income and welfare are identical to generation $T-N$.

The last element in closing the model relates to consumption after year $T$ of generations living beyond the terminal period. To capture this post-terminal consumption, we assume that consumption of these goods is such that, given the demands of the generations alive in the post-terminal years, they command a (shadow) price that declines with the rate $\bar{r}$ corresponding to a steady-state projection of the terminal year price of consumption.[7]

Appendix B outlines the general approach to formulating general equilibrium problems in the complementarity format that is used in the computer code. Appendix C presents the GAMS/MCP code for the basic pure exchange model and consists of two sub-models. The first is the calibration model discussed in Section 2.1 containing the system of equations shown in (2) to (4). The second represents the transitional dynamics and is used to solve for the effects of a change in the endowment profile.

## 2.3 Extensions

The basic exchange model can be modified in a number of directions. One possibility is to calibrate the model to an exogenous consumption profile. This involves a change in the calibration procedure. In the previous setup, the Euler equation associated with (2) determines a constant growth rate in consumption over the life cycle, $\gamma^c = c_t/c_{t-1} - 1$ ,

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steady-state path. The appropriateness of a given horizon depends on the nature of the policy scenarios. A horizon longer than 150 years may be required for extreme shocks or for policy measures which extend over several decades.

Kehoe and Levine [1985] have shown that the OLG framework may permit multiple equilibria for certain parameter values. In such cases, indeterminacy would manifest itself as sensitivity to the truncation date. None of the models presented here are sensitive to $T$, provided that it is sufficiently large. This and the general robustness of the models provides evidence that the equilibria are unique. Kotlikoff [2000] reaches the same conclusion regarding the uniqueness of equilibria in the A-K models he has been working with.

[7] Situations with an endogenous steady-state interest rate ($r_\infty$) can be handled by assigning a value based on the relative prices, i.e. $r_\infty = p_{T-1}/p_T - 1$. This would be the case if there is no possibility for international trade, if there is period by period balance of payment constrains, or if domestic and foreign goods are imperfect substitutes.

where

$$1+\gamma^c = \left(\frac{1+\bar{r}}{1+\rho}\right)^{\frac{1}{\theta}}. \tag{9}$$

Alternatively, one may choose to calibrate an OLG model to an empirically estimated life-cycle consumption profile. Changing the growth rate of consumption over the life cycle introduces $N$ constraints, one for each exogenously set value of $\gamma_t^c$. The problem therefore requires an equal number of model parameters. Letting the utility discount rate vary over time provides the required $N$ degrees of freedom. Aggregate trade balance can then be achieved by solving endogenously for $B$, the benchmark trade deficit.[8]

Extension of the model to incorporate heterogeneous households is trivial, apart from a moderate increase in dimensionality. The model presented in Appendix D has three household types, each characterized by a lifetime income profile and an intertemporal discount profile. When households are more or less patient than others, this is reflected in differences in asset accumulation and consumption profiles, as will be illustrated.

Bequests play an important part in the life-cycle income and expenditures of certain classes of households (see Kotlikoff and Summers [1981]). Bequests are important for analyzing the intergenerational impact of economic policy because they provide a mechanism through which impacts on older generations are transmitted to younger generations.

A simple means of introducing bequests to the OLG exchange model involves the introduction of a separate "bequest market" for each generation. A representative of generation $g$ then demands two *goods*, $\hat{u}_g$ and $b_g$. The first of these is the composite of lifetime consumption as defined by the original utility function in (1) and the second represents the amount of bequests to future generations. In this approach, we can think about the value of bequests as a reduced form which is calibrated to two parameters: the average bequest rate, $\beta$, and the bequest elasticity, $\xi$. If wealth of generation $g$ is $\bar{m}_g$, we have:

$$\beta = \frac{\bar{b}_g}{\bar{m}_g} \tag{10}$$

and

$$\xi = \frac{\partial b_g}{\partial m_g}\frac{m_g}{b_g}\bigg|_{\bar{m}_g,\bar{b}_g}. \tag{11}$$

In order to make the reduced form representation, we model the bequests as a specialized transfer good that is demanded by generation $g$ and caries the price $p_g^b$. Endowments of this commodity are fixed and sum to $\bar{b}_g$, hence $p_g^b$ is an index of the aggregate transfer. The ownership of the bequest good then determines which generations receive the bequest

[8] Instead of solving for $B$, one could alternatively let $\bar{r}$, $\gamma$ , or $\theta$ be endogenous. Computational problems may, however, prevent using $\theta$ if the solution implies that the value of this parameter is close to zero or unity. In applied work, selecting which parameters to endogenize is an important issue. Here one would typically want to choose the parameters that are the least certain and have the smallest impact on the results.

and $p_g^b$, equal to unity in the baseline, determines the value of the bequest.

We write the utility function for a representative generation $g$ as

$$u(b, \hat{u}) = \left[\beta^{\frac{1}{\nu}} b^{\frac{\nu-1}{\nu}} + (1-\beta)^{\frac{1}{\nu}} \hat{u}^{\frac{\nu-1}{\nu}}\right]^{\frac{\nu}{\nu-1}} \tag{12}$$

so the elasticity of substitution, $\nu$, can be used to calibrate the bequest elasticity. The "demand" for bequests arising from budget-constrained utility maximization on the baseline growth path provides the following equation:

$$\bar{b} = \frac{\beta m}{\beta p_b + (1-\beta)p_b^{\nu}}. \tag{13}$$

Evaluating $\partial m / \partial p_b$ permits us to express $\xi$ as a function of $\nu$, and we may the invert this relationship to find the elasticity of substitution as a function of the average bequest rate and the bequest elasticity:[9]

$$\nu = \frac{1 - \xi\beta}{\xi(1-\beta)}. \tag{14}$$

In order to illustrate the implementation of an OLG model with bequests, we have provided in Appendix D an exchange model with three households types: patient, impatient and wealthy. The first two households differ with respect to lifetime earnings and consumption profiles. The patient household has lower earnings early in life but greater earnings in later years than the impatient household. The two households also differ with respect to their consumption profiles – the patient household has lower consumption in early years and higher consumption in middle age. As a consequence of these assumptions, the lifetime asset profiles of these two household types are very different. The patient household accumulates assets early in life and has the greatest level of wealth around the middle of the life cycle. The impatient household borrows money early in life and spends the most of the rest of his working life paying off these debts.

The third household type, the wealthy, is identical to the patient household in every respect except that this household receives bequests from earlier generations and leaves an inheritance to later generations. We set up the model with an average bequest rate of 1% and a bequest elasticity of 2. This means that along the baseline growth path the average households leaves a bequest equal to 1% of lifetime earnings and inheritance. The bequest elasticity equal to two indicates that for every one percent reduction in income there will a two percent reduction in the level of bequest.

Figure 2 shows the asset profile over the lifetime for representative households. The patient and wealthy households are identical in terms of income and consumption, but they differ with regard to assets as a result of inheritance.

As an illustration of the impact of the bequest on model results, we perform a com-

[9]When the bequest elasticity is unity the ratio of bequest to wealth is constant, and (12) is replaced by a Cobb-Douglas function with $\beta$ as the value share of bequests.

Figure 2: Assets over the life cycle

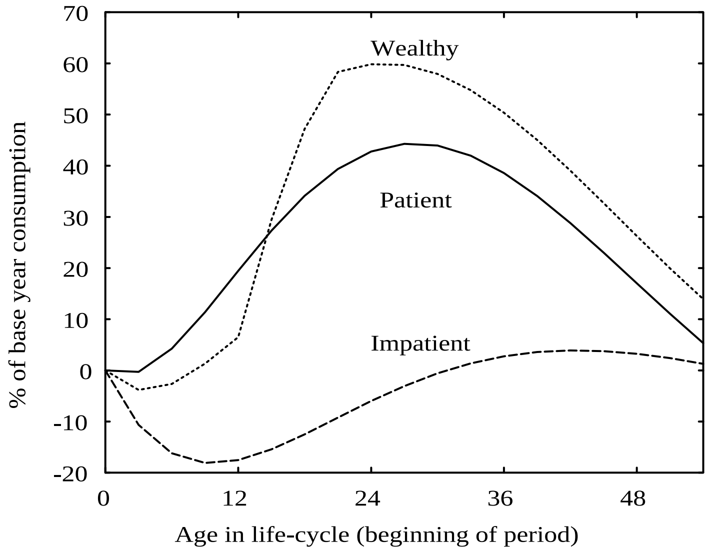
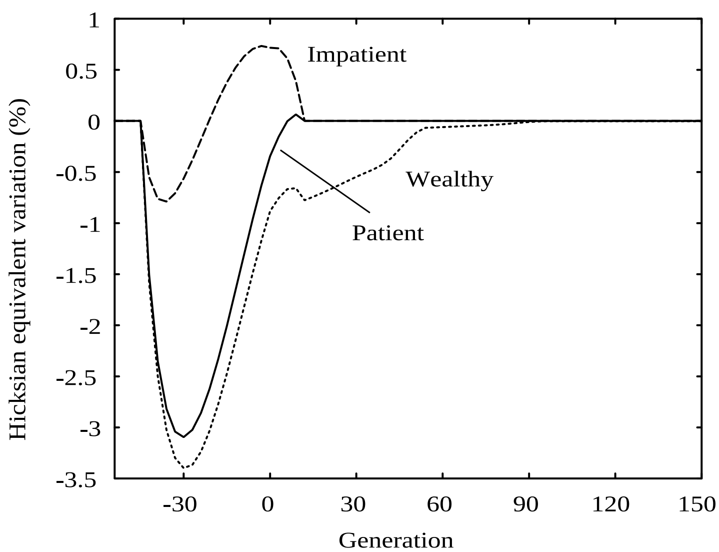



Figure 3: Welfare effects of an announced devaluation



putation in which there is an *anticipated* 25% devaluation after year 10. This shock diminishes the value of foreign assets of the older patient and rich households, who reduce consumption as a result the reduction in real wealth. As can be seen in Figure 3, the net impact of the adverse shock is positive for the impatient households who are net debtors. A comparison of welfare impacts for the patient and wealthy household illustrates the importance of bequests in the intergenerational transmission of economic impacts. The patient households born after the devaluation year 10 are completely unaffected by the devaluation, as their life-cycle incomes are identical to the baseline growth path. The impact on wealthy households extends for a number of years because the wealth reduction for the current elderly are passed forward to future generations through lower bequests.

# 3 The production model

We now expand the model of Section 2.1 to allow for production activities and a labor/leisure choice, and we show how to calibrate the model to given base year data and solve for the effects of a policy change. Including production activities is an essential part of using the model for actual policy analysis.[10] The labor/leisure choice is an important determinant of the excess burden of taxation and is therefore relevant for analyses of tax reform. In addition, including leisure in the household optimization gives rise to the possibility of corner solutions in which households may exit the workforce in latter years of the life cycle.

The calibration procedure ensures that given endowments and preferences, a solution to the household's optimization problem matches data describing the aggregate economy. We therefore proceed first by characterizing the demand side of the aggregate economy that we want the calibration procedure to replicate. We then outline the properties of the individual households' optimization problem and show how the parameters in the model can be selected to ensure that the individual and the aggregate levels are consistent. Finally, we describe the production side of the model and outline the simulation methodology for computing the effects of a policy change introduced unexpectedly in year zero. The model is calibrated to 1996 data for the U.S. The GAMS/MCP code for the production model is presented in Appendix E.

## 3.1 Aggregate demand

The demand side of our aggregate economy is characterized by national account balances relating capital income ($R$), labor income ($L$), government transfers to households ($T$), private sector consumption ($C$), private sector net saving ($S$), the primary government

[10]We employ a single sector illustrative model, the logic presented here is easily extended to a multi-sectoral framework.

budget deficit ($D$) the trade deficit ($B$), investment ($I$) and net tax rates on income from capital and labor ($\tau^r$ and $\tau^l$). These include aggregate income balance:

$$R + L + T = C + S, \tag{15}$$

savings-investment balance:

$$S - D + B = I, \tag{16}$$

and the public budget constraint:

$$\tau^r R + \tau^l L = G + T - D. \tag{17}$$

Here we omit the implicit subscript $t = 0$ on the aggregate variables when referring benchmark values. The values of these data are presented in the U.S. social accounting matrix for 1996 shown at the top of Appendix E. [11]

As in Section 2, we assume that the economy is in an initial steady state, which has the advantage of making the presentation more tractable.[12] In addition, the effects of policy changes are more transparent with a steady-state baseline as the reference case. Capital earnings and investment are linked via the capital stock which evolves according to

$$K_{t+1} = (1 - \delta) K_t + I_t, \tag{18}$$

where $\delta$ is the constant depreciation rate. The return to capital covers (after tax) interest plus depreciation on the capital stock, and investment covers growth plus depreciation. Hence the steady-state assumption implies that $R$ and $I$ are proportional:[13]

$$\frac{R}{I} = \frac{\bar{r} + \delta}{\gamma + \delta}. \tag{19}$$

We now have a complete characterization of the demand side of the aggregate economy

[11] As in Section 2, the calibration process implies deriving the baseline level of total asset holdings endogenously. Consequently, we only match a given level of earnings on the domestic capital stock, $R$. We do not calibrate to base year interest payments on foreign or government debt. As such $S$ and $D$ are net of interest payments.

[12] A non-steady-state calibration (see Wendner [1999] and Knudsen et al. [2000] ) is slightly more involved, but can be carried out by essentially the same procedure as described below. The difference is that a non-steady-state calibration requires solving the complete transition model, including all overlapping generations and the production side, to find the baseline equilibrium, while the steady-state calibration only requires knowledge of a single generation. With the present setup, both methods require solving endogenously for two of the parameters in the individuals' optimization problem to ensure that the first period solution to the model matches the benchmark data.

[13] The relationship between $R$ and $I$ implied by the steady-state assumption is not satisfied in actual data for arbitrary values of $\bar{r}$, $\delta$ , and $\gamma$. It is therefore necessary to either adjust the data or choose a combination of these parameters that satisfies ( 19). In Appendix E we adjust $I$ to be consistent with the assumed parameter values. Note that, the steady-state interest rate, $\bar{r}$, is now after tax.

and can proceed to relate it to the individual households' optimization problem.

## 3.2 Household demand

In this section we expand the household optimization problem of Section 2 to include an endogenous labor supply decision. Initial endowments $\omega_{g,t} = \omega (1+\gamma)^g$ now represent units of time rather than units of the consumption good as they did in the previous section. As such $\omega$ is an income scaling factor and is constant over the life cycle. Leisure time, $\ell_{g,t}$, enters in a constant-elasticity-of-substitution (CES) function with consumption, $c_{g,t}$, to create full consumption, $z_{g,t}$. Expressed with present value prices the optimization problem is

$$\begin{aligned} \max_{c_{g,t},\ell_{g,t}} u_{g,t}\left(z_{g,t}^{1-\theta}\right) &= \sum_{t=g}^{g+N}\left(\frac{1}{1+\rho}\right)^{t-g}\frac{z_{g,t}^{1-\theta}}{1-\theta} \\ s.t. \qquad z_{g,t} &= \left(\alpha c_{g,t}^{\sigma} + (1-\alpha)\,\ell_{g,t}^{\sigma}\right)^{\frac{1}{\sigma}} \\ \sum_{t=g}^{g+N} p_t^c c_{g,t} &\leq \sum_{t=g}^{g+N}\left[p_t^l \pi_{g,t}(\omega_{g,t}-\ell_{g,t}) + p_t^f \zeta_{g,t}\right] \\ \ell_{g,t} &\leq \omega_{g,t} \\ c_t, \ell_t &\geq 0, \end{aligned} \tag{20}$$

where $\alpha$ is the weight on consumption in full consumption, $1/(1-\sigma)$ is the elasticity of substitution between consumption and leisure, $p_t^c$ is the price of consumption, $p_t^l$ is the wage rate, $\pi_{g,t}$ is an index of productivity over the life cycle, $p_t^f = p_0^f (1+\bar{r})^{-t}$ is the price of foreign exchange, and $\zeta_{g,t}$ is a lump sum transfer from the government. [14]

As in Section 2 it is useful to characterize the outcome of the household optimization problem along the baseline. The CES demand structure implies interior solutions for $c_{g,t}$ and $\ell_{g,t}$, but may involve a corner solution in labor supply where $\ell_{g,t} = \omega_{g,t}$ in some years of the life cycle. As a result, the baseline equilibrium is fully characterized by the

[14] Aside from the inclusion of the lump-sum transfer, the setup here follows Auerbach and Kotlikoff [1987] and, for the most part, we also use their parametrization. This implies $\gamma = 0.01$ per year, $\phi = 4$, $\sigma =$ -0.25, $\pi_{g,t} = \exp\left[4.47 + 0.033\,(t-g) - 0.00067\,(t-g)^2\right]$, and that generations live for 55 years or $N = 54$. Note that it is useful to think of generations being born into the model at the time when they become economically independent at, say, 21 years. Our parametrization differs from that of Auerbach and Kotlikoff in two respects. First, we set $\delta = 0.07$ per year, whereas they do not account for capital depreciation. Second, we set $\bar{r} = 0.05$ per year and, as described below, solve endogenously for the utility discount rate finding $\rho = 0.007$, whereas they solve endogenously for the interest rate and set $\rho = 0.015$.

Government transfers, $\zeta_{g,t}$, are denominated in the price of foreign exchange, and, for simplicity, we assume that these transfers are allocated to each generation according to its share in the total population so that $\sum_g \zeta_{g,t} = T\,(1+\gamma)^t$.

following first-order conditions for generation zero:

$$
\begin{aligned}
\frac{\partial U\left(c_t, \ell_t\right)}{\partial c_t} &= \lambda \bar{p}_t && (21)\\
\frac{\partial U\left(c_t, \ell_t\right)}{\partial \ell_t} &= \eta_t && (22)\\
\eta_t - \lambda \bar{p}_t \pi_t &\geq 0 \qquad \perp \qquad \ell_t \leq \omega && (23)\\
\sum_{t=0}^{N} \bar{p}_t c_t &= \sum_{t=0}^{N} \bar{p}_t \left[\pi_t(\omega - \ell_t) + \zeta_t\right] && (24)\\
\ell_t &\leq \omega, && (25)
\end{aligned}
$$

where $\lambda$ and $\eta_t$ are, respectively, the shadow prices of the lifetime budget constraint and the time endowment in each year, and where benchmark quantities are scaled so that $\bar{p}_t = (1+\bar{r})^{-t}$ is the common present value price index in the initial steady state.

We can define the "reservation wage", i.e. the present value price of leisure, as $p^{\ell}_{g,t} = \eta_{g,t}/\lambda_g = p^l_t + \mu_{g,t}$, where $p^l_t$ is the market wage and $\mu_{g,t}$ is the excess over the market wage when $\ell_{g,t} = \omega_{gt}$. If the steady-state equilibrium, as defined above for generation zero, involves an interior solution with $\ell_t < \omega \;\; \forall t$, then, from (23), it follows that $\lambda \bar{p}_t \pi_t = \eta_t$ so that $\mu_t = 0$ and $p^{\ell}_t = p^l_t = \bar{p}_t \pi_t$. If the equilibrium involves a corner solution with $\ell_s = \omega$ in year $s$, then $\lambda \bar{p}_s \pi_s < \eta_s$ and $\mu_s > 0$ implying that the reservation wage exceeds the market wage. This turns out to be the case in the last period of the life cycle with the present parametrization, and the model thus presents an example of endogenous retirement.

## 3.3 Matching the aggregate data

For the solution to the households' maximization problem, as characterized by (21) to (25), to be consistent with the aggregate data it must be the case that summing the relevant variables at the household level over all the different generations alive in year zero gives the values in (15). In the case of labor income, $L$, and consumption, $C$, this requires:

$$
\begin{aligned}
L &= \sum_{t=0}^{N} \frac{\pi_t(\omega - \ell_t)}{(1+\gamma)^t} && (26)\\
C &= \sum_{t=0}^{N} \frac{c_t}{(1+\gamma)^t}. && (27)
\end{aligned}
$$

As in (7), consistency also requires that the sum of the value of assets equals the value of domestic equity plus the present value of future claims. In this case the aggregate value of domestic equity is positive and equal to the value of the capital stock. In addition to the effect of a permanent imbalance in the trade balance, the magnitude of total assets is also affected by the presence of the government budget deficit, $D$, implying that households at the outset must have a debt to the government thereby allowing it to run a permanent deficit:

$$\bar{A} = \sum_{t=0}^{N} m_t = (1 + \bar{r})\, K + (B - D)\, \frac{1 + \bar{r}}{\bar{r} - \gamma}, \tag{28}$$

where $m_t$ is the value of assets held in year zero by a generation of age $t$ as defined in (6), and where the quantity of capital, $K$, is multiplied by $(1 + \bar{r})$ reflecting a one-period lag in investment.

Imposing (28) as a constraint on the calibration model ensures that the total level of household assets is consistent with the capital stock, and by the relationship $R = (\bar{r} + \delta)\, K$ assets are consistent with the level of capital income. By definition $S \equiv R + L + T - C$, hence to satisfy equations (15) to (17), it is only necessary to satisfy either (26) or (27) with one implicitly insuring the other. Matching the household level with the aggregate data thus requires imposing two restrictions on the households' maximization problem and selecting two model parameters to be solved endogenously. Income is linear in $\omega$ and demand is linear in income hence $\omega$ is effectively a scaling parameter with no economic significance. This makes $\omega$ a natural candidate for one of the two calibrated parameters.[15] To see how the choice of the one remaining parameter to calibrate affects the solution, it is useful to express the Euler equation describing the growth rate of consumption over the life cycle. From (21) we get

$$1 + \bar{r} = (1 + \rho)\, (1 + \gamma_t^z)^{\theta + \sigma - 1}\, (1 + \gamma_t^c)^{1 - \sigma}\,, \tag{29}$$

where $\gamma_t^c = c_t/c_{t-1} - 1$ and $\gamma_t^z = z_t/z_{t-1} - 1$ are the growth rates of consumption and full consumption over the life cycle. Note that if the productivity profile, $\pi_t$, were flat and there is no corner solution in labor supply so that $\ell_t < \omega\ \forall t$, then $\gamma_t^z = \gamma_t^c$ and (29) simplifies to (9), the Euler equation from the exchange model.

Equation (29) shows that the growth rate of household consumption over the life cycle, and accordingly the right hand sides of (26) to (28), generally depends on the parameters $\bar{r}$, $\rho$, and $\theta$. In addition, $\delta$ also enters with $\bar{r}$ , and $\gamma$ in (19). As a result, any of these parameters may be chosen endogenously alongside $\omega$ to calibrate the model to the aggregate data.[16] In the calibration sub-model in Appendix E we add (27) and (28) to the system of complementarity equations given by (21) to (25) and solve endogenously for $\rho$ and $\omega$.

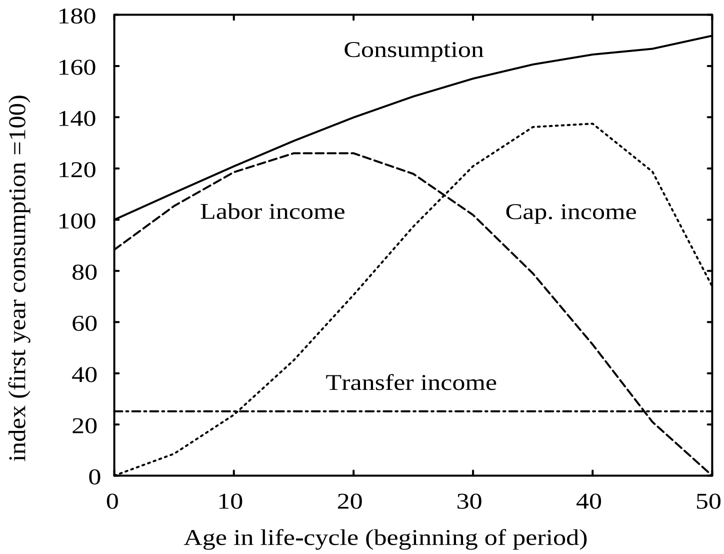
Figure 4 shows the calibrated income and consumption profiles of each generation along the baseline. The desire to increase consumption over the life cycle, as implied by (29), means that capital income is growing for the first 35 years of the life cycle and then

---

[15]Note that this implies that the model could also be calibrated by adding only one constraint and fixing either the $K/L$-ratio, or the $K/C$-ratio, and subsequently scaling all variables by the rate $\omega$.

[16]Not all combinations of $\bar{r}$, $\gamma$, $\rho$, $\theta$, $\sigma$, and $\delta$ may be feasible, however, because of non-negativity constraints.

Figure 4: Baseline income and consumption profiles




falls back toward zero reflecting positive saving while young and subsequent dissaving. Similarly, the hump-shaped productivity profile, $\pi_t$, and the tendency for leisure time to increase for a constant productivity level, as implied by the Euler equation for leisure, means that labor income is increasing for the first couple of decades of the life cycle and then decreasing. As seen from Figure 4, there is no labor supply in the last period of the life cycle with the reservation wage exceeding the market wage by 9%.

## 3.4 Production

So far we have not mentioned the production side of the economy as its structure is irrelevant for the calibration procedure. A complete specification of the economy, however, is required to calculate the effects of a policy change. The one-sector production model described below is constructed with the aim of providing the simplest possible structure consistent with the components of aggregate demand presented in Section 3.1. In particular, we assume that all markets are perfectly competitive; technology is of the CES form; trade is modeled according to the standard assumption that foreign and domestic goods are imperfect substitutes; [17] and the price of foreign goods is given on the world market according to a small open economy assumption.

[17] Without this assumption only the net trade balance would matter and the levels of imports and exports would be indeterminate.

Output, $Y_t$, is produced using inputs of labor and capital services so that

$$Y_t = \phi_Y \left\{ \beta_Y L_t^{\varepsilon} + (1 - \beta_Y) K_t^{\varepsilon} \right\}^{\frac{1}{\varepsilon}}, \tag{30}$$

where $1/(1-\varepsilon)$ is the elasticity of substitution, and where the $\phi$'s and $\beta$'s are parameters that are selected to match the baseline. Exports, $X_t$, are distinguished from output for the home market, $H_t$, by a constant elasticity of transformation function:

$$\phi_X \left[ \beta_X X_t^{\epsilon} + (1 - \beta_X) H_t^{\epsilon} \right]^{\frac{1}{\epsilon}} = Y_t, \tag{31}$$

where $1/(1+\epsilon)$ is the elasticity of transformation. Similarly, on the import side, output for the home market combines with imports, $M_t$, to produce an input composite, $A_t$, where

$$A_t = \phi_A \left[ \beta_A H_t^{\epsilon} + (1 - \beta_A) M_t^{\epsilon} \right]^{\frac{1}{\epsilon}}. \tag{32}$$

Finally, the input composite may be used for household consumption, investment, or government consumption implying the following condition for balance between aggregate supply and demand:

$$A_t = C_t + I_t + G_t. \tag{33}$$

The representation of foreign and domestic goods as imperfect substitutes has the implication that although there is a constant interest rate on the international bond market, the domestic interest rate may deviate from the world market rate during a transition period. This happens because building up the domestic capital stock requires domestic as well as imported inputs. An increase in the capital stock thereby drives up the relative price of domestic output and the domestic interest rate until the economy settles in a new steady state with constant relative prices.

## 3.5 Closing the model and the solution process

The model is closed as the exchange model in Section 2 with some modifications because post-terminal demand here involves leisure as well as consumption, and total terminal assets comprise government bonds, domestic equity and foreign assets. Post-terminal demands for leisure and consumption are treated symmetrically and as before. The main difference therefore relates to the government budget and the capital stock.

To provide a reasonable basis for welfare analysis, government consumption, like government transfers to households, is fixed at the baseline level. When evaluating the effects of a policy change, we consider balancing the government budget either period by period or over the infinite horizon, and the results may be sensitive to this assumption. In the former case, taxes are adjusted within each period so that

$$\Phi_t + p_t^f D_t = \Gamma_t, \tag{34}$$

The first term on the left equals total government tax revenue in year $t$ (i.e., $\Phi_t = \tau_t^r p_t^r R_t + \tau_T^l p_t^l L_t + \tau_t^c p_t^a C_t$). The second term is the value of the trade deficit, and the term on the right is total government expenditure ($\Gamma_t = p_t^a G_t + p_t^f T_t$). The $p$'s and $\tau$'s are the relevant prices and tax rates, and $p_t^a$ the price of the input composite from (32).

When the government budget is instead balanced over the infinite horizon, taxes are adjusted to a new constant level so that

$$p_0^f A_0^G + \sum_{t=0}^{\infty} \Phi_t = \sum_{t=0}^{\infty} \Gamma_t, \tag{35}$$

where $A_0^G = D(1+\bar{r})/(\bar{r}-\gamma)$ is the initial level of government assets which enters the model as an exogenous endowment. To represent this budget constraint in the model, it is necessary to account for the fact that a policy change may cause the government's period $T$ asset position to deviate from the baseline level. We therefore break (35) into two equations that are imposed on the model as constraints:

$$p_0^f A_0^G - p_T^f A_T^G = \sum_{t=0}^{T} (\Phi_t - \Gamma_t) \tag{36}$$

and

$$p_T^f A_T^G = \frac{1+\gamma}{\bar{r}-\gamma} (\Phi_T - \Gamma_T), \tag{37}$$

where (36) states the required balance within the model horizon as a function of the value of terminal assets, $p_T^f A_T^G$, and (37) gives the value of terminal assets as a function of income and expenditure in the terminal period using the steady-state assumption to sum over the infinite horizon.

The value of the trade deficit is similarly fixed over the infinite horizon and may deviate from the baseline level in individual periods. Instead of imposing this relationship directly, however, we require balance in all goods and transfers denominated in $p_t^f$. The present value of the trade balance is then given implicitly by the quantity of total household bond holdings in the terminal period which consists of total foreign bonds less total debt to the government. Terminal household bond holdings are selected, as in Section 2, to satisfy equation (8) so that all generations alive in the post-terminal periods achieve the same equivalent variation.[18]

The final component of aggregate assets is the domestic capital stock. Following Lau et al. [2001], the terminal capital stock is handled by requiring that investments grow at

[18]As in Section 2, first period assets are modeled as exogenous endowments. In contrast to that model, however, relative prices may now change during a transition, hence it is relevant for the results of a policy change how first period assets of the different households are divided between capital, domestic bonds and foreign bonds. Asset denomination in the terminal period is immaterial since the different types of assets in a steady-state are perfect substitutes in a deterministic world.

the steady-state rate in the last model period:

$$\frac{I_T}{I_{T-1}} = 1 + \gamma. \tag{38}$$

## 3.6 An application: fundamental tax reform

We now present an illustrative application of the model by solving for the effects of a policy change that in year zero unexpectedly and permanently reduces either the capital income tax rate or the labor income tax rate and introduces a consumption tax to balance the government budget.[19] Taxes are reduced by an amount corresponding to \$100 billion in the benchmark implying a reduction in the tax rate on capital income from 28.4% to 24.8% or a reduction in the tax rate on labor income from 42.0% to 39.2%. With no change in behavior, balancing the public budget following either tax reform would require introducing a consumption tax equal to 1.8%.

We start out by considering the case where the government budget is balanced over the infinite horizon. As seen from Figure 5, the impacts on current generations, that is generations alive in year zero, differ substantially. Reducing the capital income tax has a hump-shaped impact on current generations with a small positive impact on the old and young and a larger positive impact on the middle aged. The impact on future generations is positive and increasing until the generation entering the model 50 years after the policy change. After this time there is a small decline in the equivalent variation experienced by successive generations until it stabilizes at 0.32% on the new steady state growth path. Reducing the tax on labor income has a negative impact on the current old, whereas reducing the tax on capital income is beneficial for the young and for generations born after year zero. A labor income tax cut provides the largest improvement for generations entering the model 15 years after the policy change. Thereafter there is a small decline until the equivalent variation stabilizes at 0.17% or about half the steady state improvement associated with a reduction in the tax on capital income.

In the new steady state, where per capita welfare is constant, it seems reasonable to simply sum the equivalent variations of the different generations to measure the overall welfare change. Multiplying 0.32% and 0.17% with the cumulative value of full consumption in the benchmark shows that the long-run welfare gain from the tax reform is \$44 billion per year when the capital income tax is reduced and \$23 billion when the labor

---

[19] With five-year intervals, the present model involves 1,981 equations. A 1 GHz computer solves a counterfactual experiments in about 1 second. The PATH solution algorithm exhibits quadratic convergence, so the equilibrium values can be computed very accurately, to roughly 8 decimal places. Adopting 1-year intervals increases the number of equations to 35,821 and solution time to almost 1 minute. With larger models, the increase in solution time and reduced robustness arising from 1-year intervals may become burdensome. Longer time intervals are therefore often convenient, particularly during model development. As illustration, the model in Appendix C is based on annual time steps, Appendix D uses 3-year time steps, and Appendix E uses 5-year intervals. Converting the code from one time interval to another involves cutting and pasting 14 lines of code.

Figure 5: Welfare changes with an intertemporal budget

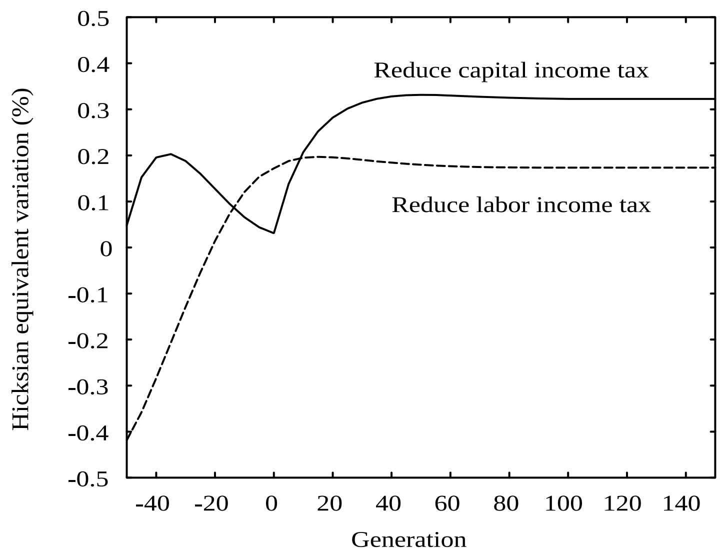
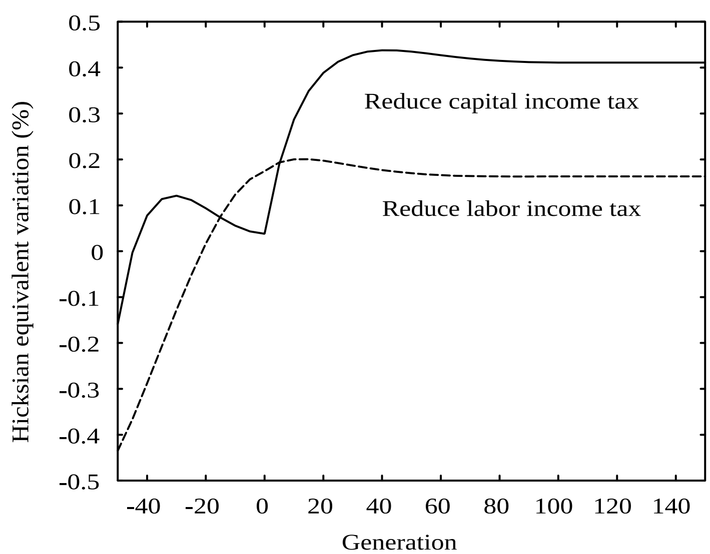



Figure 6: Welfare changes with period by period budget



income tax is reduced. These numbers compare favorably with the \$100 billion value of the tax reforms, but overestimate the true impact because they do not take into account that current generations gain much less, and some even lose. To quantify the overall welfare impact, however, requires imposing a social welfare function.

The effects of the tax reforms on current generations can be interpreted by referring to the income and consumption profiles shown in Figure 4. Shifting from a tax on capital income to a consumption tax is good for the middle aged who have large capital incomes, but it is disadvantageous for the old whose assets holdings are small relative to their consumption levels. The current young, like the current old, have relatively small assets holdings and therefore experience a negative impact on that account, but this is in their case more than offset by the gains that occur after year zero due to the positive long-run impact of the tax reform. Shifting away from the tax on labor income, in contrast, hurts both the old and the middle aged whose labor income is declining, but it benefits the young whose labor income is increasing and who gain from the long-run impact of the reform.

The effect of a tax reform on the welfare of future generations is dictated by the long-run efficiency of the new tax system as well as by intergenerational redistribution. In the baseline, the capital income tax distorts the savings decision leading to too little capital accumulation. The tax on labor income distorts the labor/leisure choice leading to under-provision of labor. The capital income tax is especially distorting in the present setting where there is no tax on interest income from foreign or government bonds. Intergenerational redistribution takes place as the policy change induces changes in consumption and income profiles over the life cycle, which cause changes in prices and tax revenue that may either hurt or benefit future generations.

Shifting from a capital income tax to a consumption tax reduces the distortion of the savings decision and results in a 5.1% increase in the capital stock in the new steady state. This causes an increase in the overall tax base so that the consumption tax need only be raised by 0.9%, substantially less than the 1.8% that would be required if there was no change in behavior. Consequently, the overall efficiency of the tax system is improved and there is also a small increase in long-run labor supply of less than 0.1%. Lowering the tax on labor income reduces the distortion of the labor/leisure choice and causes labor supply to increase by 0.2% in the long run. In this case the increase in the long-run capital stock is 0.2% and the consumption tax must be increased by 1.2% .

The consumption tax, like the labor income tax distorts the labor/leisure choice by raising the price of consumption relative to leisure. The introduction of a consumption tax, however, effectively involves a lump-sum tax on the assets of generations alive at time zero, which must be spent on consumption during their remaining lifetime. This effect means that part of the burden of financing the present value of government expenditure is shifted onto the current old to the benefit of future generations. Working in the opposite direction, the increase in the long-run tax base means that future generations are paying a relatively greater share of government expenditure. Consequently, the government budget deficit is initially falling and then increasing reflecting the shift between the two opposing effects.

To isolate the effect of intertemporal redistribution of the tax burden, we now consider the case where the government budget is balanced period by period. The reduction in the capital income tax implies a substantial increase in the tax base over time. With period by period balancing of the government budget the consumption tax rate therefore has to be increased substantially in the near term relative to the constant level implied by balancing the government budget over the infinite horizon. As shown in Figure 6, this has a substantial negative impact on the welfare changes experienced by the current old but a positive impact on future generations. With the reduction in the labor income tax, the effect on the tax base is much smaller and the two formulations of the government budget produce almost the same results.

To evaluate the overall welfare impact of moving to a consumption tax, it is useful to distinguish aggregate efficiency gains from intergenerational redistribution. The decomposition is difficult because the two effects are mutually dependent. One decomposition method, invented by Auerbach and Kotlikoff [1987], involves introducing a "Lump-Sum Redistribution Authority" (LSRA) that ensures that all generations experience the same equivalent variation. Here, each generation gives or receives a lump-sum amount of bonds to the LSRA subject to the constraint that the present value of all payments to the LSRA over the infinite future sum to zero. Using this hypothetical construct, with the government budget balanced over the infinite horizon, we find a common welfare gain of 0.16% when the capital income tax is reduced and 0.06% when the labor income tax is reduced. The cumulative value of these gains correspond to \$22 billion and \$8 billion in the benchmark. In either tax reform the common welfare gain is essentially the same whether the government budget is balanced period by period or over an infinite horizon. We therefore conclude that in either tax reform scenario roughly half of the present value increase in welfare is due to efficiency effects. Furthermore, we conclude that differences in welfare effects when the public budget is balanced period by period rather than on an infinite horizon are solely the result of intergenerational redistribution.

# 4 Concluding remarks

The overlapping generations framework has many possible applications. The obvious appeal of this framework is that it provides an important intergenerational dimension for quantifying the welfare effects of economic policies. The models presented in this paper provide a starting point for OLG modeling in the Auerbach and Kotlikoff [1987] tradition. The key contribution of the paper is to show how such models may be constructed in a systematic and computationally efficient manner.

The simple exchange model presented at the outset incorporates most of the fundamental issues involved in the overlapping generations framework. The extensions to this simple model, including bequests and multiple households may readily be incorporated into more complex models with production. As we have shown, the overlapping generations demand structure is independent of how the production side is modeled, and with the assumption of an initial steady state, the calibration procedure is also independent of the production structure. For future research, we believe that OLG modeling in a complementarity format is promising, given the simplicity of the models and the efficiency of the solution algorithms. Consequently, more features can be added to the production side, as required for the policy question at hand. Among the many possible extensions are adjustment costs in capital accumulation, intergenerational consequences of technical change, different formulations of taxation and international trade, imperfect competition and endogenous growth.

# References


**Altig, David, Alan Auerbach, Laurence Kotlikoff, Kent Smetters, and Jan Walliser**, "Simulating Fundamental Tax Reform in the United States," *American Economic Review*, 2001. Forthcoming.

—— **and Charles T. Carlstrom**, "Marginal Tax Rates and Income Inequality in a Life-Cycle Model," *American Economic Review*, 1999, *89* (5), 1197–1215.

**Auerbach, Alan J. and Laurence J. Kotlikoff**, *Dynamic Fiscal Policy*, Cambridge, MA: Cambridge University Press, 1987.

**Blanchard, Olivier J.**, "Debt, Deficits, and Finite Horizons," *Journal of Political Economy*, 1985, *93* (2), 223–247.

**Broer, D. Peter, Ed W. M. T. Westerhout, and A. Lans Bovenberg**, "Taxation, Pensions, and Saving in a Small Open Economy," *Scandinavian Journal of Economics*, 1994, *96* (3), 403– 424.

**Brooke, Anthony, David Kendrick, Alexander Meeraus, and Ramesh Raman**, *GAMS: A Users Guide,* GAMS Development Corporation, 1998.

**Doquier, Frédéric and Philippe Liégeois**, "Simulating Computable Overlapping Generations Models With TROLL," *Computational Economics*, 2001. Forthcoming.

**Ferris, Michael C. and Todd. S. Munson**, "Complementarity Problems in GAMS and the PATH Solver," *Journal of Economic Dynamics and Control*, 2000, *24* (2), 165–188.

**Ferris, Micheal C., Olvi L. Mangasarian, and Jong-Shi S. Pang, eds**, *Complementarity: Applications, Algorithms and Extensions*, Dordrecht, The Netherlands: Kluwer Academic Publishers, 2001.

**Fullerton, Don and Diane Lim Rogers**, *Who Bears the Lifetime Tax Burden*, Washington, DC: The Brookings Institution, 1993.

**Hubbard, Glenn R., Jonathan Skinner, and Stephen P. Zeldes**, "Why do People Save," *American Economic Review*, 1994, *84* (2), 174–179.

**Kehoe, Timothy J. and David K. Levine**, "Comparative Statics and Perfect Foresight in Infinite Horizon Economies," *Econometrica*, 1985, *53* (2), 433–453.

**Keuschnigg, Christian and Wilhelm Kohler**, "Modeling Intertemporal General Equilibrium: An Application to Austrian Commercial Policy," *Empirical Economics*, 1994, *19* (1), 131–164.

**Knudsen, Martin B, Lars Haagen Petersen, Toke Ward Petersen, Peter Stephensen, and Peter Trier**, "A Dynamic CGE Analysis of the Danish Aging Problem," in Glenn W. Harrison, Svend E. H. Jensen, Lars H. Pedersen, and Thomas F. Rutherford, eds., *Using Dynamic General Equilibrium Modeling for Policy Analysis*, Amsterdam: North-Holland, 2000.

**Kotlikoff, Laurence J.**, "The A-K OLG Model: Its Past Present and Future," in Glenn W. Harrison, Svend E. H. Jensen, Lars H. Pedersen, and Thomas F. Rutherford, eds., *Using Dynamic General Equilibrium Models for Policy Analysis*, Amsterdam: North-Holland, 2000.

—— **and Lawrence H. Summers**, "The Role of Intergenerational Transfers in Aggregate Capital Accumulation," *Journal of Political Economy*, 1981, *89* (4), 706–732.

**Kotlikoff, Laurence, Smetters Kent A., and Jan Walliser**, "Privatizing Social Security in the U.S.: Comparing the Options," *Review of Economic Dynamics*, 1999, *2* (3), 532–574.

—— **,** —— **, and** —— , "Distributional Effects in a General Equilibrium Analysis of Social Security," in Martin Feldstein, ed., *The Distributional Effects of Social Security Reform*, Chicago: University of Chicago Press, 2001. Forthcoming.

**Lau, Morten I.**, "Assessing Tax Reforms when Human Capital is Endogenous," in "Using Dynamic General Equilibrium Models for Policy Analysis," Amsterdam: North-Holland, 2000.

—— **, Andreas Pahlke, and Thomas F. Rutherford**, "Approximating Infinite-Horizon Models in a Complementarity Format: A Primer in Dynamic General Equilibrium Analysis," *Journal of Economic Dynamics and Control*, 2001. Forthcoming.

**Miles, David**, "Modelling the Impact of Demographic Change upon the Economy," *The Economic Journal*, 1999, *109* (1), 1–37.

**Rutherford, Thomas F.**, "Extensions of GAMS for Complementarity Problems Arising in Applied Economics," *Journal of Economic Dynamics and Control*, 1995, *19* (8), 1299–1324.

—— , "Applied General Equilibrium Modeling with MPSGE as a GAMS Subsystem: An Overview of the Modeling Framework and Syntax," *Computational Economics*, 1999, *14*, 1–46.

**Wendner, Ronald**, "A Calibration Procedure of Dynamic CGE Models for Non-Steady State Situations Using GEMPACK," *Computational Economics*, 1999, *13* (3), 265–287.

———, "An Applied Dynamic General Equilibrium Model of Environmental Tax Reforms and Pension Policy," *Journal of Policy Modeling*, 2001. Forthcoming.

# Appendix A

This appendix demonstrates the relationship between the initial value of assets in the economy and the trade deficit along the steady state. Substituting for $m_a$ in equation (7) by using (5) and (6), we have

$$\overline{A} = \sum_{t'=0}^{N} \left( \frac{1+\bar{r}}{1+\gamma} \right)^{t'} \left[ \sum_{t=0}^{t'-1} \frac{\omega_t - c_t}{(1+\bar{r})^t} \right].$$

Interchanging the summation sequence, we obtain

$$\overline{A} = \sum_{t=0}^{N} \frac{(\omega_t - c_t)}{(1+\bar{r})^t} \left[ \sum_{t'=t+1}^{N} \left( \frac{1+\bar{r}}{1+\gamma} \right)^{t'} \right].$$

We can then evaluate the geometric series over $t'$ explicitly:

$$\overline{A} = \sum_{t=0}^{N} \frac{\omega_t - c_t}{(1+\gamma)^t} (1+\bar{r}) \left[ \frac{1 - \left( \frac{1+\bar{r}}{1+\gamma} \right)^{N-t}}{\gamma - \bar{r}} \right],$$

which can then be written as:

$$\overline{A} = \frac{1+\bar{r}}{\bar{r}-\gamma} \left[ \left( \frac{1+\bar{r}}{1+\gamma} \right)^{N} \sum_{t=0}^{N} \frac{\omega_t - c_t}{(1+\bar{r})^t} - \sum_{t=0}^{N} \frac{\omega_t - c_t}{(1+\gamma)^t} \right].$$

The first summation within the square brackets is equal to zero because the steady-state growth path satisfies the income balance condition in (3). From (4) the second summation within the square brackets is equal to $-B$, the period 0 trade surplus. Hence

$$\overline{A} = \frac{1+\bar{r}}{\bar{r}-\gamma} B = \sum_{t=0}^{\infty} \left( \frac{1+\gamma}{1+\bar{r}} \right)^{t} B.$$

# Appendix B

This appendix outlines how the models appearing Appendix C to E are formulated according to the GAMS/MPSGE convention. In this approach, four classes of equilibrium conditions characterize an economic equilibrium:

1. Zero profit conditions, identified in the GAMS/MCP code by equation names with the prefix "PRF". All constant-returns-to-scale production activities earn zero excess profit in equilibrium, and this zero profit condition exhibits complementary slackness with respect to the associated activity level. For production activity $j$, there is an associated unit revenue function $R_j(p)$ and unit cost function $C_j(p)$. We can define the unit profit function as $\Pi_j(p) = R_j(p) - C_j(p)$. The associated equilibrium conditions are:
$$-\Pi_j(p) \geq 0, \quad y_j \geq 0, \quad y_j \Pi_j(p) = 0$$

2. Market clearance conditions, identified in the GAMS/MCP code by equation names beginning "MKT". Supply must be greater than or equal to demand for each primary factor and produced good, and these inequalities must exhibit complementary slackness with respect to market prices. Exploiting Shepard's lemma, the net supply of goods and factors from the production activities can be expressed using gradients of the profit functions.[20] Given a household endowment matrix $\omega_{ih}$ and household demand functions $d_{ih}(p, M_h)$:
$$\xi_i \equiv \sum_j y_j \frac{\partial \Pi_j(p)}{\partial p_i} + \sum_h \omega_{ih} - \sum_h d_{ih}(p, M_h) \geq 0, \quad p_i \geq 0, \quad p_i \xi_i = 0$$

3. Income definitions, identified in the GAMS/MCP code by equation names beginning "DEF". In equilibrium, the income for household $h$ equals the value of factor endowment at equilibrium prices:
$$M_h = \sum_i p_i \omega_{ih}$$

4. Auxiliary equations, identified in the GAMS/MCP code by equation names beginning "EQU". These equations characterize model closure rules such as the price of consumption after the model horizon or transversality conditions which characterize asset holdings in the terminal period by governments or generations living beyond the terminal period.

[20] When there are tax distortions affecting factor choice, Shephard's lemma implies that factor demands represent gradients with respect to the user cost, gross of tax.

When writing these systems of equations, there are methods of representing the cost, revenue and production netput terms which define functions relative to a benchmark equilibrium point. For example, consider a production technology in which the unit cost function has a CES form:

$$C(p) = \phi \left( \sum_i \alpha_i p_i^{1-\sigma} \right)^{\frac{1}{1-\sigma}},$$

and the revenue function is a constant-elasticity-of-transformation (CET) form:

$$R(p) = \psi \left( \sum_i \beta_i p_i^{1+\eta} \right)^{\frac{1}{1+\eta}}.$$

In this case, the netput vector, which can be computed as the gradient of the profit function, has the form:

$$\frac{\partial \Pi(p)}{\partial p_i} = \beta_i \left( \frac{p_i}{R} \right)^{\eta} - \alpha_i \left( \frac{C}{p_i} \right)^{\sigma}.$$

The values of $\alpha_i$ and $\beta_i$ define the technology, yet they are not directly observed in the benchmark equilibrium. Instead, a reference equilibrium point provides four types of data which locally characterize a production function: (i) reference input prices ($\overline{p}_i{}^I$), (ii) reference output prices ($\overline{p}_i{}^O$), (iii) reference inputs ($\overline{x}_i{}^I$) and (iv) reference outputs ($\overline{x}\,{}^O$).

Define $\overline{C}$ as the benchmark cost and $\overline{R}$ as the benchmark revenue. (Re)define $\alpha_i$ as the *benchmark input value share* of good $i$, and (re)define $\beta_i$ as the *benchmark output value share* of good $i$; i.e.,

$$\begin{aligned} \overline{C} &= \sum_i \overline{p}_i{}^I \overline{x}_i{}^I, \\ \overline{R} &= \sum_i \overline{p}_i{}^O \overline{x}_i{}^O, \\ \alpha_i &= \overline{p}_i{}^I \overline{x}_i{}^I / \overline{C}, \end{aligned}$$

and

$$\beta_i = \overline{p}_i{}^O \overline{x}_i{}^O / \overline{R}.$$

Then the profit can be written as

$$\Pi(p) = \overline{R} \left( \sum_i \beta_i \left( p_i / \overline{p}_i{}^O \right)^{1+\eta} \right)^{\frac{1}{1+\eta}} - \overline{C} \left( \sum_i \alpha_i \left( p_i / \overline{p}_i{}^I \right)^{1-\sigma} \right)^{\frac{1}{1-\sigma}},$$

and the netput vector is

$$\frac{\partial \Pi(p)}{\partial p_i} = \overline{x}_i{}^O \left( \frac{p_i / \overline{p}_i{}^O}{R(p)/\overline{R}} \right)^{\eta} - \overline{x}_i{}^I \left( \frac{C(p)/\overline{C}}{p_i / \overline{p}_i{}^I} \right)^{\sigma}.$$

In some production functions, such as in sectors where there is only a single output, the functional form for $R(p)$ may be very simple, e.g., $R(p) = p_j$. *If we know that such a sector must operate at positive intensity in equilibrium*, then the input demand functions may be written more compactly by exploiting the fact that in equilibrium $C(p)/\overline{C} = p_j/\overline{p}_j^{\,O}$. In this case, the compensated demand function for good $i$ can be written as a sparse function of only two goods, $p_i$ and $p_j$:

$$\frac{\partial C(p)}{\partial p_i} = \overline{x}_i^{\,I} \left( \frac{p_j/\overline{p}_j^{\,O}}{p_i/\overline{p}_i^{\,I}} \right)^{\sigma} .$$

# Appendix C: An OLG exchange model

```
$TITLE Appendix C: OLG exchange model -- constant discount rate

*=======================================================================
* Introduce intertemporal sets
*=======================================================================

* The model captures all generations alive in the first model period
* (year 0) and all those born in the span of the subsequent 150
* years, where generations are labeled according to the year in which
* they are born. The model is solved in 1-year intervals with each
* new generation being born at the start of a period and living to
* the age of 55.

SCALARS
        TIMINT          Single period time interval            /1/,
        INIYEAR         Year in with oldest generation was born /-54/;
SETS
        G               Generations in the model               /
        "-54","-53","-52","-51",
        "-50","-49","-48","-47","-46","-45","-44","-43","-42","-41",
        "-40","-39","-38","-37","-36","-35","-34","-33","-32","-31",
        "-30","-29","-28","-27","-26","-25","-24","-23","-22","-21",
        "-20","-19","-18","-17","-16","-15","-14","-13","-12","-11",
        "-10","-9","-8","-7","-6","-5","-4","-3","-2","-1", 0*150/,
        T(G)            Time periods in the model         /0*150/,
        A(T)            Typical life-cycle /0*54/;

* We need some special sets to identify key time periods and generations.

SETS    TFIRST(T)       First period in the model,
        TLAST(T)        Last period in the model,
        ATGEN(G)        Generations with terminal assets;

* These special sets are identified by their order in the declaration.

TFIRST(A)       = YES$(ORD(A) EQ 1);
TLAST(T)        = YES$(ORD(T) EQ CARD(T));
ATGEN(G)        = YES$((CARD(G)-CARD(A)+1) LT ORD(G));

* Aliases used to manipulate sets.

ALIAS (G,GG,YR), (T,TT), (A,AA);

*=======================================================================
* Introduce fundamental parameters
*=======================================================================
```

```
SCALARS RBAR_A          Annual interest rate                    /0.05/,
        GAMMA_A         Annual population growth rate           /0.01/,
        THETA           Exponent in intertemporal utility       /4.00/,
        TDEF0           Base year trade deficit                 /0.01/;

* Modify annual rates of change to 5-year interval between solution
* periods

SCALARS RBAR            Periodic interest rate,
        GAMMA           Periodic population growth rate;

RBAR    = (1+RBAR_A)**TIMINT  - 1;
GAMMA   = (1+GAMMA_A)**TIMINT - 1;

*======================================================================
* Time profiles
*======================================================================

* Declare variables relating values to intertemporal sets and use
* annual growth and interest rates to create time profiles consistent
* with the 1-year interval between solution periods.

PARAMETERS
        YEAR(G)         Point in time,
        AGE(A)          Age at a given point in the life cycle,
        PREF(G)         Reference price path (present value price index),
        QREF(G)         Reference quantity path (population size index),
        OMEGA0(A)       Initial endowment profile;

* Time periods and ages can be identified by the order of the
* relevant set using the fact that each period has a length of 5
* years, and that generations are labeled according to the year they
* are born and live for 55 years.

YEAR(G)         = INIYEAR + TIMINT * (ORD(G)-1);
AGE(A)          = TIMINT * (ORD(A)-1);

* Declare indices for population size and present value prices.

QREF(G)         = (1+GAMMA_A)**YEAR(G);
PREF(G)         = 1 /(1+RBAR_A)**YEAR(G);

* Use ages and years to set up correspondence between generations,
* age, and year.

SET     MAPG(G,A,YR) Assignment from generation and age to time period;

MAPG(G,A,YR) = YES$(YEAR(G)+AGE(A) EQ YEAR(YR));
```

```
* Endowment profile as in Auerbach and Kotlikoff (1987):

OMEGA0(A) = EXP(4.47 + 0.033*AGE(A) - 0.00067 * AGE(A)**2);

* Endowment profiles are scaled to an economy-wide level of 1 in the
* base year

OMEGA0(A) = OMEGA0(A) / SUM(AA, (1+GAMMA)**(1-ORD(AA)) * OMEGA0(AA));

*======================================================================
* Calibration model: Solve for benchmark steady state of reference
* generation, with calibration model for endogenous consumption
*======================================================================

* Find utility discount rate, RHO, to set implied aggregate
* consumption equal to aggregate endowments plus the trade deficit.
* We use the equations arising from the household utility
* maximization problem to set up a mixed complementarity problem
* (MCP) and use the solver to find the value of RHO that satisfies
* all the equations in the system. Alternatively, since the
* first-order conditions in this simple setting dictate that
* consumption is growing over the life cycle at a constant rate
* one could solve for rho analytically as it is done in appendixd.gms

VARIABLES
        RHO     Period utility discount rate;

POSITIVE VARIABLES
        CC(T)   Consumption profile of generation born in year 0,
        LAMDA   Shadow price of income (present value utils);

EQUATIONS
        EQC(A)  First order condition for consumption,
        EQCCC   Base year aggregate consumption,
        EQLAMDA First order condition for price of income;

* First order conditions.

EQC(A)..        LAMDA*PREF(A) =E= (1+RHO)**(1-ORD(A))*CC(A)**(-THETA);
EQLAMDA..       SUM(A,PREF(A)*CC(A)) =E= SUM(A,PREF(A)*OMEGA0(A));

* Aggregate consumption in the base year is implied by the
* consumption profile of the generation born in year 0 since the
* steady state assumption determines the relative sizes of each
* generation. We require that aggregate consumption equals total
* endowments plus the trade deficit.

EQCCC..          SUM(A,CC(A)/QREF(A)) =E= 1 + TDEF0;
```

```
* Associate variables with equations.

MODEL BENCH     /EQC.CC, EQLAMDA.LAMDA, EQCCC.RHO/;

* Initialize variables and set bounds to prevent operation errors.

RHO.L           = 0.01;
RHO.LO          = -0.99;
CC.L(A)         = 1/CARD(A);
CC.LO(A)        = 1E-9;
LAMDA.L         = 1;
LAMDA.LO        = 1E-9;

* Solve calibration model.

BENCH.ITERLIM=50000;
SOLVE BENCH USING MCP;

* Derived utility discount rate on an annual basis.

PARAMETER RHO_A   Annual utility discount (%);

RHO_A           = 100 * ((1+RHO.L)**(1/TIMINT) - 1); DISPLAY RHO_A;

*======================================================================
* Use endowments and the calibrated consumption profile for
* generation 0 to install baseline values for all generations
*======================================================================

PARAMETERS
        EREF(G,T)       Baseline endowment profile,
        CREF(G,T)       Baseline consumption profile,
        PREF_T(A)       Baseline post-terminal price path,
        CREF_T(G,A)     Baseline post-terminal consumption profile,
        MREF(G)         Baseline present value of consumption;

* We assign demand and income profiles for generation G at time T
* based on endowments and the calibrated consumption profile for
* generation 0. The trick here is to use a GAMS loop over the mapping
* which relates generations (G), ages (A) and time periods (T):

LOOP(MAPG(G,A,T),
        EREF(G,T) = QREF(G) * OMEGA0(A);
        CREF(G,T) = QREF(G) * CC.L(A);  );

* The last model generation is born in year 150 which means that in
* order to capture the full life cycle of all model generations we
* need to cover a 50-year "post-terminal" period. We index these
* post-terminal periods by the same index (A) that we use to index
```

```
* ages in a life cycle.

* Present value prices in post-terminal periods are extrapolated from
* the value of the reference price index in the terminal period.

LOOP((A,TLAST)$AGE(A), PREF_T(A) = PREF(TLAST) / (1 + RBAR_A)**AGE(A); );

* Consumption profiles in post-terminal periods for generation G at
* age AA are inferred from the consumption levels in the initial
* period of generations that have the same age.

LOOP((G,A,TLAST)$(YEAR(G)+AGE(A) GT YEAR(TLAST)),
        CREF_T(G,AA)$(AGE(AA)+YEAR(TLAST) EQ AGE(A)+YEAR(G))
        = CC.L(A) * QREF(G); );

* Present value of consumption by generation, including post-terminal
* consumption by generations who live beyond the model horizon.

MREF(G) = SUM(T, PREF(T)*CREF(G,T)) + SUM(A,  PREF_T(A)*CREF_T(G,A));

*======================================================================
* Use endowments and the calibrated consumption profiles for
* generation 0 to back out the evolution of asset holdings and
* distinguish between domestic and foreign debt
*======================================================================

PARAMETERS
        ASSETS(A)       Present value of assets over the life cycle,
        MA(A)           Assets held in year 0 by age of generation,
        ASSETH          Positive asset holdings by age in year 0,
        DEBT            Net debt by age in year 0;

SCALAR  THETAD          Ratio of total assets to total debt;

* The present value of assets equals the sum of the value of
* endowments less consumption in all previous periods of the life
* cycle. The asset profile of the representative generation can then
* be used to find the distribution of asset holdings across
* generations alive in the base year.

ASSETS(A) = SUM(AA$(ORD(AA) LT ORD(A)), PREF(AA)*(OMEGA0(AA) - CC.L(AA)));
MA(A)     = ASSETS(A)/(QREF(A)*PREF(A));

* We assume that negative asset positions reflect holdings of both
* domestic and foreign debt, while positive asset positions reflect
* holdings of domestic assets. We assume that all age groups with
* negative assets hold foreign and domestic debt in the same
* proportion which means that we can use the ratio of total assets to
* total debt to decompose the asset holdings by type
```

```
THETAD         = -SUM(A$(MA(A) GT 0), MA(A))/ SUM(A$(MA(A) LT 0), MA(A));

ASSETH(A,"DOMESTIC")$(MA(A) GT 0)       =  MA(A);
DEBT(A,"DOMESTIC")$(MA(A) LT 0)         = -MA(A)*THETAD;
DEBT(A,"FOREIGN")$(MA(A) LT 0)          = -MA(A)*(1-THETAD);

* We model baseline assets and debt positions as inital endowments
* for generations alive in year 0.

PARAMETERS
        DASSET(G)       Initial endowment of domestic assets,
        FASSET(G)       Initial endowment of foreign assets;

DASSET(G) = SUM(MAPG(G,A,"0"), ASSETH(A,"DOMESTIC") - DEBT(A,"DOMESTIC"));
FASSET(G) = SUM(MAPG(G,A,"0"), - DEBT(A,"FOREIGN"));

*======================================================================
* Model in GAMS/MCP. This model solves for the equilibrium transition
* path subject to terminal conditions that assume the presence of a
* steady state. If there are no exogenous changes the model
* replicates the calibrated consumption profiles. We use this feature
* to check the calibration and then solve for the results of en
* exogenous change in the endowment profile
*======================================================================

POSITIVE VARIABLES

* Production activities. These determine how inputs are converted
* into outputs according to the technology implied by the benchmark
* data. The variables here are activity levels and an equilibrium
* requires that each active sector earns zero profit.

        U(G)            Utility,
        X(T)            Export,
        M(T)            Import,

* Prices. The variables here are the prices that are associated with
* each commodity. An equilibrium requires that prices are such that
* supply equals demand.

        PC(T)           Price of private consumption,
        PCT(G,A)        Price of post-terminal consumption of goods,
        PU(G)           Price of intertemporal utility,
        PFX             Price of foreign exchange,

* Consumer income levels. These are agents that receive income from
* endowments or taxes and spend it to maximize utility. The variables
* here are income levels and an equilibrium requires that total
```

```
* income equals total expenditure.

        RA(G)           Representative agents by generation

* These are endogenous variables associated with model constraints
* that relate the transition to the steady state.

        CT(G,A)         Post-terminal consumption of goods,
        AT(G)           Terminal assets;

* Equations asscociated with model variables. These fall into three
* classes. PRF which ensure zero profit in each activity, MKT which
* ensure no excess demand for each commodity, and DEF which ensure
* income balance for each agent.

EQUATIONS
        PRF_U(G)        Utility,
        PRF_X(T)        Export,
        PRF_M(T)        Import,
        MKT_PC(T)       Price of private consumption,
        MKT_PCT(G,A)    Price of post-terminal consumption of goods,
        MKT_PU(G)       Price of intertemporal utility,
        MKT_PFX         Price of foreign exchange,
        DEF_RA(G)       Representative agents by generation,
        EQU_CT(G,A)     Post-terminal consumption of goods,
        EQU_AT(G)       Terminal assets;

* Utility is treated of as a commodity demanded by the different
* generations which implies that the utility function is modeled as
* any other production activity. The activity level here is initialized
* at unity implying an overall output level equal to the present value
* of consumption, MREF(G).

PRF_U(G)..      SUM(T, PREF(T)   * CREF(G,T) * (PC(T)/PREF(T))**(1-1/THETA))
                + SUM(A, PREF_T(A) * CREF_T(G,A) *
                (PCT(G,A)/PREF_T(A))**(1-1/THETA))
                =E= MREF(G) * PU(G)**(1-1/THETA);

* These equations represent zero profit in import and export activities.
* The assumption of a small open economy and perfect capital mobility
* implies that the price of imports is constant in current value terms.
* We therefore do not need to distinguish between years, but only
* operate with a single present value price for foreign exchange. The
* import activity is initialized at the reference quantity path implying
* an overall output level equal to the baseline trade deficit while the
* export activity is initialized at zero.

PRF_M(T)..      PFX * PREF(T) =G= PC(T);
```

```
PRF_X(T)..      PC(T) =G= PFX * PREF(T);

* Supply equals demand for consumption, post-terminal consumption,
* "utility", and foreign exchange.

MKT_PC(T)..     SUM(G, EREF(G,T) + DASSET(G)$TFIRST(T)) + M(T) - X(T) =E=


                SUM(G, CREF(G,T)*U(G)*(PU(G)*PREF(T)/PC(T))**(1/THETA));

MKT_PCT(G,A)$CREF_T(G,A)..
                CT(G,A) =E= U(G)*(PU(G)*PREF_T(A)/PCT(G,A))**(1/THETA);

MKT_PU(G)..     U(G)*MREF(G)*PU(G) =E= RA(G);

MKT_PFX..       SUM(T, X(T)*PREF(T)) + SUM(G, FASSET(G))
                =E= SUM(T, M(T)*PREF(T));

* Income balance for each generation. Each generation demands "utility"
* and is endowed with an amount of the consumption good in each period.
* In addition, generations alive in the initial period are endowed with
* domestic and foreign assets. To terminate the model generations alive
* in the terminal period are required to leave an amount of assets and
* are also endowed with goods for consumption in the post-terminal periods.

DEF_RA(G)..     RA(G) =E= SUM(T, PC(T)*(EREF(G,T) + DASSET(G)$TFIRST(T)))
                + PFX * FASSET(G) + (PFX * AT(G))$ATGEN(G)
                + SUM(A, PCT(G,A) * CREF_T(G,A) * CT(G,A));

* Select the levels of post-terminal consumption so that the present
* value price declines with the steady state interest rate.

EQU_CT(G,A)$CREF_T(G,A)..
                SUM(TLAST, PC(TLAST)) =E= PCT(G,A)*(1+RBAR)**(ORD(A)-1);

* Select terminal asset position so that all generations living past
* the terminal period achieve the same equivalent variation.

EQU_AT(G)$ATGEN(G)..
                U(G) =E= U(G-1);

* Define the equations entering the model and their complementary
* slackness relationship with variables in the model:

MODEL EXCHANGE /PRF_U.U, PRF_X.X, PRF_M.M , MKT_PC.PC, MKT_PCT.PCT,
                MKT_PU.PU, MKT_PFX.PFX, DEF_RA.RA ,EQU_CT.CT, EQU_AT.AT/;

* Assign initial values for those variables which are not assigned
* explicitly below:
```

```
*======================================================================
* Assign initial values and bounds for activity levels, prices, and
* auxiliary variables
*======================================================================

U.L(G)          = 1;
PU.L(G)         = 1;
PFX.L           = 1;
RA.L(G)         = MREF(G);
X.L(T)          = 0;
M.L(T)          = QREF(T)*TDEF0;
PC.L(T)         = PREF(T);
PCT.L(G,A)      = PREF_T(A);
CT.L(G,A)       = 1$CREF_T(G,A);
AT.L(G)         = SUM(T, (CREF(G,T) -EREF(G,T)) * PREF(T))$ATGEN(G);
AT.LO(G)        = -INF;

* Numeraire:

PC.FX(TFIRST)   = 1;

*======================================================================
* Replicate the benchmark equilibrium
*======================================================================

EXCHANGE.ITERLIM=0;
SOLVE EXCHANGE USING MCP;
DISPLAY "Benchmark tolerance CHK:",EXCHANGE.OBJVAL;

*======================================================================
* Run counterfactual
*======================================================================

* Modify the endowment profile, and scale it to maintain an economy-wide
* level of unity in the base year.

PARAMETER       OMEGA(A)        Counterfactual endowment profile;

OMEGA(A)        = EXP(4.47 + 0.02*AGE(A) - 0.0007*(AGE(A))**2);
OMEGA(A)        = OMEGA(A) / SUM(AA, (1+GAMMA)**(1-ORD(AA)) * OMEGA(AA));

LOOP(A,EREF(G,T)$MAPG(G,A,T) = QREF(G) * OMEGA(A); );

* Solve the model.

EXCHANGE.ITERLIM=10000;
SOLVE EXCHANGE USING MCP;
```

```
* Parameters for reporting results from the counterfactual experiment.

PARAMETERS
        WCHANGE Welfare change (% equivalent variation by year of birth),
        TDEF    Trade deficit (change from baseline level);

* Due to production activities being homogenous of degree 1 the
* equivalent variation is the percentage change in output from sector
* U. The trade deficit in year T is the difference between aggregate
* consumption and aggregate endowment.

WCHANGE(G)      = 100 * (U.L(G) - 1);
TDEF(T)         = M.L(T) - X.L(T);

*======================================================================
* Display statements
*======================================================================

DISPLAY YEAR, AGE, PREF, PREF_T, QREF, OMEGA0, OMEGA, EREF, CREF, CREF_T,
        MREF, ASSETS, MA, ASSETH, DEBT, DASSET, FASSET, WCHANGE, TDEF;
```

# Appendix D: An OLG bequest model

```
$TITLE Appendix D: OLG exchange --  multiple households and bequests

*========================================================================
* Introduce intertemporal sets
*========================================================================

* The model captures all generations alive in the first model period
* (year 0) and all those born in the span of the subsequent 150 years,
* where generations are labeled according to the year in which they are
* born. The model is solved in 3-year intervals with each new generation
* being born at the start of a period and living to the age of 57.

SCALARS TIMINT          Single period time interval             /3/,
        INIYEAR         Year in with oldest generation was born /-54/;

SETS    G               Generations in the model
        /"-54","-51","-48","-45","-42","-39","-36","-33","-30",
        "-27","-24","-21","-18","-15","-12","-9","-6","-3",
        0,3,6,9,12,15,18,21,24,27,30,33,36,39,42,45,48,51,54,57,
        60,63,66,69,72,75,78,81,84,87,90,93,96,99,
        102,105,108,111,114,117,120,123,126,129,132,
        135,138,141,144,147,150/,
        T(G) Time periods in the model /
        0,3,6,9,12,15,18,21,24,27,30,33,36,39,42,45,48,51,54,57,
        60,63,66,69,72,75,78,81,84,87,90,93,96,99,102,105,108,111,
        114,117,120,123,126,129,132,135,138,141,144,147,150/,
        A(T) Life-cycle
        /0,3,6,9,12,15,18,21,24,27,30,33,36,39,42,45,48,51,54/;

* We need some special sets to identify key time periods and
* generations.

SETS     TFIRST(T)       First period in the model,
         TLAST(T)        Last period in the model,
         ATGEN(G)        Generations with terminal assets;

* These special sets are identified by their order in the
* declaration.

TFIRST(A)       = YES$(ORD(A) EQ 1);
TLAST(T)        = YES$(ORD(T) EQ CARD(T));
ATGEN(G)        = YES$((CARD(G)-CARD(A)+1) LT ORD(G));

* There are three types of households:

SET     H       Households /PATIENT, IMPATIENT, WEALTHY/;
```

```
* Aliases used to manipulate sets.

ALIAS (G,GG,YR), (T,TT), (A,AA), (H,HH);

*======================================================================
* Introduce fundamental parameters
*======================================================================

SCALARS RBAR_A          Annual interest rate                    /0.05/,
        GAMMA_A         Annual population growth rate           /0.01/,
        THETA           Exponent in intertemporal utility       /4.00/;

* Modify annual rates of change to the time interval between
* solution periods:

SCALARS RBAR            Periodic interest rate,
        GAMMA           Periodic population growth rate;

RBAR    = (1+RBAR_A)**TIMINT  - 1;
GAMMA   = (1+GAMMA_A)**TIMINT - 1;

PARAMETER       NUMBER(H)       Relative numbers of households
                                /WEALTHY 1, PATIENT 2, IMPATIENT 10/;

SCALAR  CAD0    Current account deficit (calibrated);

*======================================================================
* Time profiles
*======================================================================

* Declare parameters relating values to intertemporal sets and use
* annual growth and interest rates to create time profiles consistent
* with the interval between solution periods.

PARAMETERS
        YEAR(G)         Point in time,
        AGE(A)          Age at a given point in the life cycle,
        PREF(G)         Reference price path (present value price index),
        QREF(G)         Reference quantity path (index of population size),
        CONSUM(A,H)     Consumption profile
        BQDISTR(A)      Bequest distribution
        ENDOW(A,H)      Endowment profile
        BEQUEST(G,GG,H) Bequest (present value) by household H generation G,
        BQR(A,H)        Period 0 bequest receipts
        BQV(H)          Bequestion value for household H
        PVBQ(G,H)       Present value bequest
        BQN(H)          Net bequests received by period 0 generation
        ESUBB(G,H)      Top-level elasticity bw consumption and bequests,
        BETA_B(A,H)     Average bequest as fraction of residual income
```

```
        BETA(H)         Average bequest (fraction of wealth)  /WEALTHY 0.01/
        XI(H)           Bequest elasticity wrt income /WEALTHY 2/;

* Initialize these parameters to 0:

BEQUEST(G,GG,H) = 0;
ESUBB(G,H)      = 0;

* Time periods and ages can be identified by the order of the relevant
* set using the fact that each period has a length of 5 years, and that
* generations are labeled according to the year they are born and live
* for 55 years.

YEAR(G)         = INIYEAR + TIMINT * (ORD(G)-1);
AGE(A)          = TIMINT * (ORD(A)-1);

SCALAR LIFESPAN;  LIFESPAN = SMAX(A, YEAR(A)) - YEAR("0");

* Declare indices for population size and present value prices.

PREF(G)         = 1 /(1+RBAR_A)**YEAR(G);
QREF(G)         = (1+GAMMA_A)**YEAR(G);

* Use ages and years to set up correspondence from generations, life
* cycle and year:

SET     MAPG(G,A,YR) Assignment from generation and age to time period;

MAPG(G,A,YR)    = YES$(YEAR(G)+AGE(A) EQ YEAR(YR));

* Endowment profile for rich as in Auerbach and Kotlikoff (1987):

ENDOW(A,"WEALTHY") = EXP(4.47 + 0.033*AGE(A) - 0.00067 * AGE(A)**2);
ENDOW(A,"PATIENT") = ENDOW(A,"WEALTHY");

* Assume a "flatter" endowment profile for the impatient reflecting a
* lower rate of human capital formation in early years:

ENDOW(A,"IMPATIENT") = SQRT(ENDOW(A,"PATIENT"));

* Endowment profiles are scaled to an economy-wide level of 1 in the
* base year.

ENDOW(A,H) = NUMBER(H) * ENDOW(A,H)
           / SUM((AA,HH),
           (1+GAMMA)**(1-ORD(AA)) * NUMBER(HH) * ENDOW(AA,HH));

* Define bequest distribution. This is the fraction of a $ of bequests
* in a given year recieved by genereations as defined by their age in
```

```
* that year. It is assumed that only those between the age of 10 and 20
* recieve bequests according to the following formula.

BQDISTR(A)$((10 LE AGE(A)) AND (AGE(A) LE 20))
        = 0.14 + AGE(A)*0.038 + AGE(A)*AGE(A)*(-0.002);

* Scale these to sum to unity.

BQDISTR(A) = BQDISTR(A) / SUM(AA, BQDISTR(AA));

* Compute the base year bequest. The bequest given by households in
* generation 0 is related to labor income and the level of bequests
* they receive during the their lifetime.

BQV(H)  = BETA(H) * SUM(A, PREF(A)*ENDOW(A,H))
        / (1-BETA(H)*SUM(A,BQDISTR(A)*PREF(A)*QREF(A)));

* Compute beqeust receipts for the generations alive in period 0.
* Bequests are distributed in the final period of economic activity for
* a given generation, so period 0 bequests are distributed by the
* oldest generation.

BQR(A,H) = BQDISTR(A) * BQV(H) * ((1+RBAR)/(1+GAMMA))**(CARD(A)-1);

* Net bequest receipts for generation 0 households.

BQN(H) = SUM(A, BQR(A,H)*QREF(A)*PREF(A)) - BQV(H);

* Infer present value of bequest receipts to generation G from
* generation GG for all generations from the bequest receipts in year 0.

LOOP((G,GG,A,TLAST)$((YEAR(G)+LIFESPAN EQ YEAR(GG)+AGE(A))
        AND (YEAR(G)+LIFESPAN LE YEAR(TLAST))),
        BEQUEST(G,GG,H)$BETA(H) = BQDISTR(A)*BQV(H)*QREF(G)*PREF(G););

* The present value bequest by generation G is the sum of all bequests
* to generations GG.

PVBQ(G,H) = SUM(GG, BEQUEST(G,GG,H));

* Impose exogenous consumption profiles.

CONSUM(A,"WEALTHY")    = 1 +
        AGE(A)*(1.2620/1E2 + AGE(A) * (4.8180/1E4 + AGE(A) * (-9.5569)/1E6));
CONSUM(A,"PATIENT")    =  CONSUM(A,"WEALTHY");
CONSUM(A,"IMPATIENT")  =  1;

* Scale consumption for budget balance.
```

```
CONSUM(A,H)     = CONSUM(A,H)*(SUM(AA, ENDOW(AA,H)*PREF(AA))+BQN(H))
                / SUM(AA, CONSUM(AA,H)*PREF(AA));

* Compute bequest as shares of remaining life time income for
* generations in year 0.

BETA_B(A,H) = BQV(H) / (BQV(H)
         + SUM(AA$(AGE(AA) GE AGE(A)), PREF(AA)*CONSUM(AA,H)));

* Compute elasticity of substitution between top level consumption
* index and bequests according to formula.

ESUBB(G,H)$XI(H) = (1-BETA(H)*XI(H)) / (XI(H)*(1-BETA(H)));

LOOP(MAPG(G,A,TFIRST),
        ESUBB(G,H)$XI(H) = (1-BETA(H)*XI(H)) / (XI(H)*(1-BETA(H)));)

* Calibrate the current account deficit as a residual:

CADO    = SUM((A,H), (CONSUM(A,H)-ENDOW(A,H))/QREF(A));

*======================================================================
* Use endowments and the calibrated consumption profiles for generation
* 0 to to back out the evolution of asset holdings and distinguish
* between domestic and foreign debt
*======================================================================

PARAMETERS
        ASSETS(A,H)     Present value of assets over the lifecycle,
        MA(A,H)         Current assets by age in period 0,
        ASSETH          Positive asset holdings by age in period 0,
        DEBT            Net Debt by age in period 0;

SCALARS
        DEFICIT         Gross deficit among domestic households,
        SURPLUS         Gross surplus among domestic households,
        THETAD          Domestic share of domestic debt         /1/,
        THETAA          Domestic share of domestic assets       /1/;

* The present value of assets equals the sum of the value of endowments
* less consumption in all previous periods of the life cycle. The asset
* profile of the representative generation can then be used to find the
* distribution of asset holdings accross generations alive in the base
* year.

ASSETS(A,H)     = SUM(AA$(ORD(AA) LT ORD(A)), PREF(AA)
                * (ENDOW(AA,H) + QREF(AA)*BQR(AA,H) - CONSUM(AA,H)));
MA(A,H)         = ASSETS(A,H)/(QREF(A)*PREF(A));
```

```
* We assume that negative asset positions reflect holdings of both
* domestic and foreign debt, while positive asset positions reflect
* holdings of domestic assets. We assume that all age groups with
* negative assets hold foreign and domestic debt in the same proportion
* which means that we can use the ratio of total assets to total debt
* to decompose the asset holdings by type.

DEFICIT         = -SUM((A,H)$(MA(A,H) LT 0), MA(A,H));
SURPLUS         =  SUM((A,H)$(MA(A,H) GT 0), MA(A,H));

THETAD$DEFICIT  = MIN(1, SURPLUS / DEFICIT);
THETAA$SURPLUS  = MIN(1, DEFICIT / SURPLUS);

ASSETH(A,H,"DOMESTIC")$(MA(A,H) GT 0) = MA(A,H)  * THETAA;
ASSETH(A,H,"FOREIGN")$(MA(A,H) GT 0)  = MA(A,H)  * (1-THETAA);

DEBT(A,H,"DOMESTIC")$(MA(A,H) LT 0)   = -MA(A,H) * THETAD;
DEBT(A,H,"FOREIGN")$(MA(A,H) LT 0)    = -MA(A,H) * (1-THETAD);

* We model baseline assets and debt positions as inital endowments
* for generations alive in year 0.

PARAMETERS      DASSET(*,H)     Domestic asset initial endowments
                FASSET(*,H)     Foreign asset intiial endowments;

DASSET(G,H)     = SUM(MAPG(G,A,"0"),
                  ASSETH(A,H,"DOMESTIC") - DEBT(A,H,"DOMESTIC"));
FASSET(G,H)     = SUM(MAPG(G,A,"0"),
                  ASSETH(A,H,"FOREIGN")  - DEBT(A,H,"FOREIGN"));
DASSET("TOTAL",H) = SUM(G, DASSET(G,H));
FASSET("TOTAL",H) = SUM(G, FASSET(G,H));

*======================================================================
* Use solution for reference generation to install baseline values for
* all generations
*======================================================================

PARAMETERS
        EREF(G,H,T)     Baseline endowment profile,
        CREF(G,H,T)     Baseline consumption profile,
        BREF(G,H,T)     Time path of bequest receipts,
        PREF_T(A)       Baseline post-terminal price path,
        CREF_T(G,H,A)   Baseline post-terminal consumption profile,
        MREF(G,H)       Baseline present value of consumption,
        RAREF(G,H)      Baseline aggregate income including bequest;

* We assign demand and income profiles for generation G at time T based
* on endowments and the calibrated consumption profile for generation
* 0. The trick here is to use a GAMS loop over the mapping which relates
```

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* generations (G), ages (A) and time periods (T):

LOOP(MAPG(G,A,T),
        EREF(G,H,T) = QREF(G) * ENDOW(A,H);
        BREF(G,H,T) = QREF(T) * BQR(A,H);
        CREF(G,H,T) = QREF(G) * CONSUM(A,H);  );

* The last model generation is born in year 150 which means that in
* order to capture the full life cycle of all model generations we need
* to cover a 50-year "post-terminal" period. We index these
* post-terminal periods by the same index (A) that we use to index ages
* in a life cycle.

* Present value prices in post-terminal periods are extrapolated from
* the value of the reference price index in the terminal period.

LOOP((A,TLAST)$AGE(A),  PREF_T(A) = PREF(TLAST) / (1 + RBAR_A)**AGE(A); );

* Consumption profiles in post-terminal periods for generation G at age
* AA are inferred from the consumption levels in the initial period of
* generations that have the same age.

LOOP((G,A,TLAST)$(YEAR(G)+AGE(A) GT YEAR(TLAST)),
        CREF_T(G,H,AA)$(AGE(AA) + YEAR(TLAST) EQ AGE(A) + YEAR(G))
        = CONSUM(A,H) * QREF(G); );

* Present value of consumption by generation, including post-terminal
* consumption by generations who live beyond the model horizon.

MREF(G,H)       = SUM(T, PREF(T)*CREF(G,H,T))
                + SUM(A, PREF_T(A)*CREF_T(G,H,A));
RAREF(G,H)      = MREF(G,H) + SUM(GG, BEQUEST(G,GG,H));

*=====================================================================
* Import and export price levels for counterfactual
*=====================================================================

PARAMETERS
        PX(T)   Export price,
        PM(T)   Import price;

PX(T) = 1;
PM(T) = 1;

POSITIVE VARIABLES

* Declare production activities. These determine how inputs are
* converted into outputs according to the technology implied by the
* benchmark data. The variables here are activity levels and an
```

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* equilibrium requires that each active sector earns zero profit.

        U(G,H)        Utility
        X(T)          Export
        M(T)          Import

* Declare prices. The variables here are the prices that are associated
* with each commodity. An equilibrium requires that prices are such
* that supply equals demand.

        PC(T)         Price of private consumption,
        PCT(G,H,A)    Price of post-terminal consumption of goods,
        PU(G,H)       Price of intertemporal utility,
        PRA(G,H)      Price index over consumption and bequest,
        PB(G,H)       Bequest made by generation G,
        PFX           Price of foreign exchange,

* Income variables. These are agents that receive income from
* endowments or taxes and spend it to maximize utility. The variables
* here are income levels and an equilibrium requires that total income
* equals total expenditure.

        RA(G,H)       Representative agents by generation

* Variables associated with model constraints that relate the
* transition to the steady state.

        CT(G,H,A)     Post-terminal consumption of goods
        AT(G,H)       Terminal assets;

* Equations asscociated with model variables. These fall into three
* classes. PRF which ensure zero profit in each activity, MKT which
* ensure no excess demand for each commodity, and DEF which ensure
* income balance for each agent.

EQUATIONS

        PRF_U(G,H)      Utility,
        PRF_X(T)        Export,
        PRF_M(T)        Import,
        MKT_PC(T)       Price of private consumption,
        MKT_PCT(G,H,A)  Price of post-terminal consumption of goods,
        MKT_PU(G,H)     Price of intertemporal utility,
        MKT_PB(G,H)     Bequest made by generation G,
        MKT_PFX         Price of foreign exchange,
        DEF_RA(G,H)     Representative agents by generation,
        DEF_PRA(G,H)    Price index over consumption and bequest,
        EQ_CT(G,H,A)    Post-terminal consumption of goods,
        EQ_AT(G,H)      Terminal assets ;
```

```
* Utility is treated of as a commodity demanded by the different
* generations which implies that the utility fuction is modeled as any
* other production activity.  The activity level here is initialized at
* unity implying an overall output level equal to the present value
* of consumption, MREF(G).

PRF_U(G,H)..
        SUM(T, PREF(T)   * CREF(G,H,T) * (PC(T)/PREF(T))**(1-1/THETA)) +
        SUM(A, PREF_T(A) * CREF_T(G,H,A) *
       (PCT(G,H,A)/PREF_T(A))**(1-1/THETA))
        =E= MREF(G,H) * PU(G,H)**(1-1/THETA);

* These equations represent zero profit in import and export activities.
* The assumption of a small open economy and perfect capital mobility
* implies that the price of imports is constant in current value terms.
* We therefore do not need to distinguish between years, but only
* operate with a single present value price for foreign exchange. The
* import activity is initialized at the reference quantity path implying
* an overall output level equal to the baseline trade deficit while the
* export activity is initialized at zero.

PRF_X(T)..      PC(T) =G= PFX * PREF(T) * PX(T);

PRF_M(T)..      PFX * PREF(T) * PM(T) =G= PC(T);

* Supply equals demand for consumption, post-terminal consumption,
* and foreign exchange.

MKT_PC(T)..     SUM((G,H), EREF(G,H,T) + DASSET(G,H)$TFIRST(T)) + M(T)
                =E= SUM((G,H), CREF(G,H,T) * U(G,H) *
                (PU(G,H)*PREF(T)/PC(T))**(1/THETA)) + X(T);

MKT_PCT(G,H,A)$CREF_T(G,H,A)..
                CT(G,H,A) =E= U(G,H) * (PU(G,H)*PREF_T(A)/PCT(G,H,A))**(1/THETA);

MKT_PFX..       SUM(T, X(T)*PREF(T)*PX(T)) + SUM((G,H), FASSET(G,H))
                + SUM((G,H)$ATGEN(G), AT(G,H))
                =E= SUM(T, M(T)*PREF(T)*PM(T));

* Income balance for each generation of type H. Each household demands
* "utility" and is endowed with bequests and an amount of the consumption
* good in each period. In addition, generations alive in the initial
* period are endowed with domestic and foreign assets. To terminate the
* model, generations alive in the terminal period are required to leave
* an amount of assets and are also endowed with goods for consumption in
* the post-terminal periods.

DEF_RA(G,H)..   RA(G,H) =E= SUM(T, PC(T) * (EREF(G,H,T)
```

```
                + DASSET(G,H)$TFIRST(T)))
                + SUM(GG, PB(GG,H)*BEQUEST(GG,G,H))
                + PFX * FASSET(G,H) + (PFX * AT(G,H))$ATGEN(G)
                + SUM(A, PCT(G,H,A) * CREF_T(G,H,A) * CT(G,H,A));

* Define price index over consumption and bequest

DEF_PRA(G,H)..  ( (MREF(G,H)/RAREF(G,H)) * PU(G,H)**(1-ESUBB(G,H))
                + (PVBQ(G,H)/RAREF(G,H)) * PB(G,H)**(1-ESUBB(G,H))
                - PRA(G,H)**(1-ESUBB(G,H)) )$(ABS(1-ESUBB(G,H)) GT 0.01)
                +
                ( PU(G,H)**(MREF(G,H)/RAREF(G,H))
                * PB(G,H)**(PVBQ(G,H)/RAREF(G,H))
                - PRA(G,H) )$(ABS(1-ESUBB(G,H)) LE 0.01) =E= 0;

* Supply demand balance in utility market

MKT_PU(G,H)..   U(G,H) =E= (RA(G,H)/(RAREF(G,H)*PRA(G,H)))
                * (PRA(G,H)/PU(G,H))**ESUBB(G,H);

* Supply demand balance in bequest market

MKT_PB(G,H)$PVBQ(G,H)..
                PVBQ(G,H) =E=
                PVBQ(G,H)*(RA(G,H)/(RAREF(G,H)*PRA(G,H)))
                * (PRA(G,H)/PB(G,H))**ESUBB(G,H);

* Select terminal asset position so that all generations living past
* the terminal period achieve the same equivalent variation.

EQ_AT(G,H)$ATGEN(G)..  U(G,H) - U(G-1,H) =E= 0;

* Select the levels of post-terminal consumption so that the present
* value price declines with the steady state interest rate.

EQ_CT(G,H,A)$CREF_T(G,H,A)..
                SUM(TLAST, PC(TLAST)) =E= PCT(G,H,A)*(1+RBAR)**(ORD(A)-1);

* Define the equations entering the model and their complementary
* slackness relationship with variables in the model:

MODEL EXCHANGE /PRF_U.U,PRF_X.X,PRF_M.M,
                MKT_PC.PC,MKT_PCT.PCT,MKT_PU.PU,MKT_PB.PB,
                MKT_PFX.PFX,DEF_RA.RA,DEF_PRA.PRA,EQ_CT.CT,EQ_AT.AT /;

*======================================================================
* Assign initial values for activity levels, prices, and auxiliary
* variables:
*======================================================================
```

```
U.L(G,H)         = 1;
PU.L(G,H)        = 1;
PRA.L(G,H)       = 1;
PFX.L            = 1;
PB.L(G,H)        = 1;
RA.L(G,H)        = RAREF(G,H);
X.L(T)           = 0;
M.L(T)           = QREF(T)*CAD0;
PC.L(T)          = PREF(T);
PCT.L(G,H,A)     = PREF_T(A);
CT.L(G,H,A)      = 1$CREF_T(G,H,A);
AT.L(G,H)$ATGEN(G) = SUM(T, CREF(G,H,T)*PREF(T))
                   - SUM(T, (EREF(G,H,T)+BREF(G,H,T))*PREF(T));
AT.LO(G,H)       = -INF;

* Numeraire:

PC.FX(TFIRST)    = 1;

*======================================================================
* Replicate the benchmark equilibrium:
*======================================================================

EXCHANGE.WORKSPACE=20;
EXCHANGE.ITERLIM=0;
SOLVE EXCHANGE USING MCP;
DISPLAY "Benchmark tolerance CHK:",EXCHANGE.OBJVAL;

* Parameters for reporting results from counterfactual experiment:

PARAMETER
  WCHANGE        Welfare change (% equivalent variation by year of birth),
  CADEF          Current account deficit (% of benchmark consumption)
  C(G,H,T)       Consumption levels;

C(G,H,T) = CREF(G,H,T)*U.L(G,H)*(PU.L(G,H)*PREF(T)/PC.L(T))**(1/THETA);
CADEF(T,"BMK") =
        100 * SUM((G,H),(C(G,H,T)-EREF(G,H,T)))/SUM((G,H),C(G,H,T));

*======================================================================
* Run counterfactual:
*======================================================================

* Change the exchange rate as reflected in the price of imports and
* exports. Initial holdings of foreign assets are fixed so this
* represents a sudden fall in the price of foreign assets which helps
* the poor who have negative assets, but hurts the rich.
```

```
PM(T)$(YEAR(T) GT 10) = 1.25;
PX(T)$(YEAR(T) GT 10) = 1.25;

OPTION SYSOUT=ON;
EXCHANGE.ITERLIM=10000;
SOLVE EXCHANGE USING MCP;

* Compute welfare changes and current account deficit:

C(G,H,T) = CREF(G,H,T)*U.L(G,H)*(PU.L(G,H)*PREF(T)/PC.L(T))**(1/THETA);
WCHANGE(G,H)            = 100 * (U.L(G,H) - 1);
CADEF(T,"bequest")      = 100 * SUM((G,H), (C(G,H,T)-EREF(G,H,T)))
                        / SUM((G,H),C(G,H,T));

*======================================================================
* Display statements
*======================================================================

DISPLAY YEAR,AGE,PREF,QREF,ATGEN,BQV,BQR,BQN,BEQUEST,PVBQ,ESUBB,CAD0;
DISPLAY EREF,CREF,CREF_T,MREF,ASSETH,DEBT,DASSET,FASSET;
DISPLAY WCHANGE, CADEF, ASSETS;
```

# Appendix E: An OLG production model

```
$TITLE Appendix E: OLG production model -- period by period govt budget

$ONTEXT

The benchmark social accounting matrix:

      | Output| Income categories    |  Consumption categories
      | OUT   | CAP     LAB     TAX  |  CON     INV     GOV     ROW
----------------------------------------------------------------------
OUT   |                              5,397   1,786   1,474     802
CAP   | 3,521
LAB   | 5,041
TAX   |           779   1,491
CON   |         2,742   3,550                          995
INV   |                              1,890            -199      95
GOV   |                         2,270
ROW   |   897
----------------------------------------------------------------------
Note: Based on 1996 IO tables for USA. Numbers in 1996 USD billion.

$OFFTEXT

*=======================================================================
* Introduce intertemporal sets
*=======================================================================

* The model captures all generations alive in the first model period
* (year 0) and all those born in the span of the subsequent 150
* years, where generations are labeled according to the year in which
* they are born. The model is solved in 5-year intervals with each
* new generation being born at the start of a period and living to
* the age of 55.

SCALARS TIMINT  Single period time interval                     /5/,
        INIYEAR Year in with oldest generation was born         /-50/;

SETS
        G Generations in the model /
        "-50","-45","-40","-35","-30","-25","-20","-15","-10","-5",
        0,5,10,15,20,25,30,35,40,45,50,55,60,65,70,75,80,85,90,95,
        100,105,110,115,120,125,130,135,140,145,150/,
        T(G) Time periods in the model /
        0,5,10,15,20,25,30,35,40,45,50,55,60,65,70,75,80,85,90,95,
        100,105,110,115,120,125,130,135,140,145,150/,
        A(T)   Typical life-cycle / 0,5,10,15,20,25,30,35,40,45,50/;

* We need some special sets to identify key time periods and generations.
```

```
SETS    TFIRST(T)       First period in the model,
        TLAST(T)        Last period in the model,
        ATGEN(G)        Generations with terminal assets;

* These special sets are identified by their order in the declaration.

TFIRST(A)       = YES$(ORD(A) EQ 1);
TLAST(T)        = YES$(ORD(T) EQ CARD(T));
ATGEN(G)        = YES$((CARD(G)-CARD(A)+1) LT ORD(G));

* Aliases used to manipulate sets.

ALIAS (G,GG,YR), (T,TT), (A,AA);

*======================================================================
* Introduce fundamental parameters
*======================================================================

SCALARS RBAR_A          Annual interest rate                    /0.05/,
        GAMMA_A         Annual population growth rate           /0.01/,
        DELTA_A         Annual depreciation rate                /0.07/,
        THETA           Inverse intertemporal elasticity        /4.00/,
        SIGMA_CL        Elasticity of substitution (C vs L)     /0.8/,
        PHI             Consumption share parameter             /0.4/,
        ETADX           Elasticity of transformation D vs. X    /4/,
        SIGMA           Armington elasticity on imports         /4/;

* Modify annual rates of change to 5-year interval between solution
* periods

SCALARS RBAR            Periodic interest rate,
        GAMMA           Periodic population growth rate,
        DELTA           Periodic depreciation rate,
        RHO_CL          Exponent in intratemporal utility;

RBAR    = (1+RBAR_A)**TIMINT  - 1;
GAMMA   = (1+GAMMA_A)**TIMINT - 1;
DELTA   = 1 - (1-DELTA_A)**TIMINT;
RHO_CL  = 1 - 1/SIGMA_CL;

*======================================================================
* Time profiles
*======================================================================

* Declare variables relating values to intertemporal sets and use
* annual growth and interest rates to create time profiles consistent
* with the 5-year interval between solution periods.
```

```
PARAMETERS
        YEAR(G)         Point in time,
        AGE(A)          Age at a given point in the life cycle,
        PREF(G)         Reference price path (present value index),
        QREF(G)         Reference quantity path (index),
        PSHR(A)         Population share for agents of age A,
        PI(A)           Productivity index;

* Time periods and ages can be identified by the order of the
* relevant set using the fact that each period has a length of 5
* years, and that generations are labeled according to the year they
* are born and live for 55 years.

YEAR(G)         = INIYEAR + TIMINT * (ORD(G)-1);
AGE(A)          = TIMINT * (ORD(A)-1);

* Declare indices for population size and present value prices.

QREF(G)         = (1+GAMMA_A)**YEAR(G);
PREF(G)         = 1 /(1+RBAR_A)**YEAR(G);

* Age group A share in total polutation

PSHR(A) = (1/QREF(A)) / SUM(AA, (1/QREF(AA)));

* Productivity index as in Auerbach and Kotlikoff (1987)

PI(A) = EXP(4.47 + 0.033*AGE(A) - 0.00067*(AGE(A))**2) / EXP(4.47);

* Use ages and years to set up correspondence between generations,
* age, and year.

SET MAPG(G,A,YR)    Assignment from generation and age to time period;

MAPG(G,A,YR)    = YES$(YEAR(G)+AGE(A) EQ YEAR(YR));

*======================================================================
* Read benchmark data
*======================================================================

SCALARS C0      Benchmark private consumption          /5.397/,
        I0      Benchmark investment                   /1.786/,
        G0      Benchmark government consumption       /1.474/,
        X0      Benchmark exports                      /0.802/,
        M0      Benchmark imports                      /0.897/,
        R0      Benchmark capital earnings (net of tax) /2.742/,
        L0      Benchmark labor earnings (net of tax)   /3.550/,
        S0      Benchmark savings                      /1.890/,
        T0      Benchmark transfers  to households     /0.995/,
```

```
        D0      Benchmark government budget deficit     /0.199/,
        B0      Benchmark trade deficit                 /0.095/,
        Y0      Benchmark output                        /8.562/,
        TR0     Benchmark tax rate on capital income
        TL0     Benchmark tax rate on labor income;

TR0     = 779  / 2742;
TL0     = 1491 / 3550;

* Infer capital stock from earnings and the steady-state return:

SCALAR        K0      Initial capital stock;

K0      = R0 / (RBAR+DELTA);

*======================================================================
* Modify benchmark data to represent a consistent steady-state
*======================================================================

* Modify investment level and revise consumption accordingly to keep
* total demand constant:

C0      = C0 + I0 - (GAMMA+DELTA)*K0;
I0      = (GAMMA+DELTA)*K0;
S0      = R0 + L0 + T0 - C0;

*======================================================================
* Calibration model: Solve for benchmark steady state of reference
* generation
*======================================================================

* Find utility discount rate, RHO, and time endowment, OMEGA, to set
* implied aggregate values at the benchmark level. Government
* transfers to households, T0, are modeled as exogenous lump-sum
* payments and are added to household incomes according to each
* generation's share in the total population. We use the equations
* arising from the household utility maximization problem to set up
* an mixed complementarity problem (MCP) and use the solver to find
* the value of RHO that satisfies all the equations in the system.

VARIABLES
        CA(A)           Present value of assets over the life cycle,
        CMA(A)          Value of assets held by age,
        CCMA            Aggregate value of assets held by age,
        RHO             Period utility discount rate,
        OMEGA           Scaling factor on time endowment;

POSITIVE VARIABLES
        CZ(A)           Full consumption,
```

```
        CC(A)           Consumption,
        CCC             Aggregate consumption,
        CELL(A)         Leisure time,
        CL(A)           Labor time,
        CPZ(A)          Price of full consumption (present value),
        ETA(A)          Price of time (present value utils),
        LAMDA           Price of income (present value utils);

EQUATIONS
        EQZ(A)          Definition of consumption,
        EQPZ(A)         Definition of price of full consumption,
        EQC(A)          FOC for consumption,
        EQS(A)          FOC for leisure time,
        EQL(A)          FOC for labor time,
        EQETA(A)        FOC for price of time,
        EQLAMDA         FOC for price of income,
        EQCA(A)         Present value assets over the life cycle,
        EQCMA(A)        Value of assets held by age,
        EQCCC           Aggregate consumption,
        EQCCMA          Aggregate value of assets,
        EQCCCLEV        Fix level of aggregate consumption,
        EQCCMALEV       Fix level of aggregate assets;

* Equation definitions:

EQZ(A)..        CZ(A) =E= ( PHI*CC(A)**RHO_CL
                +(1-PHI)*CELL(A)**RHO_CL )**(1/RHO_CL);

* First order conditions:

EQC(A)..        LAMDA*PREF(A) =E= (1+RHO)**(1-ORD(A))
                * CZ(A)**(1-RHO_CL-THETA)
                * PHI*CC(A)**(RHO_CL-1);

EQS(A)..        ETA(A) =E= (1+RHO)**(1-ORD(A))
                * CZ(A)**(1-RHO_CL-THETA)
                * (1-PHI)*CELL(A)**(RHO_CL-1);

EQL(A)..        ETA(A)=G=LAMDA*PREF(A)*PI(A);

EQLAMDA..        SUM(A,PREF(A)*CC(A)) =E=
                SUM(A,PREF(A)*(PI(A)*CL(A) + PSHR(A)*QREF(A)*T0));

EQETA(A)..      OMEGA =E= CELL(A) + CL(A);

* Price indices:

EQPZ(A)..       CPZ(A) =E=
                (PHI**SIGMA_CL*PREF(A)**(1-SIGMA_CL) + (1-PHI)**SIGMA_CL
```

```
                * (ETA(A)/LAMDA)**(1-SIGMA_CL))**(1/(1-SIGMA_CL));

* Asset positions:

EQCA(A)..        CA(A) =E= SUM(AA$(ORD(AA) LT ORD(A)),
                 PREF(AA)*(PI(AA)*CL(AA)
                 + PSHR(AA)*QREF(AA)*T0-CC(AA)));

EQCMA(A)..      CMA(A) =E= CA(A)/(QREF(A)*PREF(A));

* Aggregate values:

EQCCC..         CCC =E= SUM(A,CC(A)/QREF(A));
EQCCMA..        CCMA =E= SUM(A,CMA(A));

* Fix aggregate level of consumption and assets at benchmark level:

EQCCCLEV..      CCC =E= C0;
EQCCMALEV..     CCMA =E= (1+RBAR)*K0 + (B0-D0)*(1+RBAR)/(RBAR-GAMMA);

* Associate variables with equations:

MODEL BENCH     /EQZ.CZ, EQPZ.CPZ, EQCA.CA, EQCMA.CMA, EQC.CC,
                EQS.CELL,EQL.CL, EQETA.ETA, EQLAMDA.LAMDA, EQCCC.CCC,
                EQCCMA.CCMA, EQCCCLEV.RHO, EQCCMALEV.OMEGA/;

* Set bounds to prevent operation errors:

RHO.LO          = -0.99;
CZ.LO(A)        = 1E-5;
CC.LO(A)        = 1E-5;
CELL.LO(A)      = 1E-5;
LAMDA.LO        = 1E-5;

* Initialize variables:

RHO.L           = 0.01;
CZ.L(A)         = 0.5;
CC.L(A)         = 0.5;
CELL.L(A)       = 0.5;
CL.L(A)         = 0.5;
CPZ.L(A)        = 0.5;
ETA.L(A)        = 0.5;
LAMDA.L         = 0.5;
CCC.L           = 0.5;

* Solve calibration model:

BENCH.ITERLIM=50000;
```

```
SOLVE BENCH USING MCP;

* Calibration results

PARAMETERS      RHO_A           Annual utility discount rate,
                PELLRATIO(A)    Ratio of reservation to market wage;

RHO_A           = (1+RHO.L)**(1/TIMINT) - 1;
PELLRATIO(A)    = 100 * ETA.L(A)/(PREF(A)*PI(A)*LAMDA.L);

*======================================================================
* Use endowments and the calibrated consumption and leisure time
* profiles for generation 0 to install baseline values for all
* generations
*======================================================================

PARAMETERS
        PIREF(G,T)      Baseline productivity profile,
        EREF(G,T)       Baseline endowment profile,
        CREF(G,T)       Baseline consumption profile,
        ELLREF(G,T)     Baseline leisure time profile,
        LREF(G,T)       Baseline labor time profile,
        ZREF(G,T)       Baseline full consumption profile,
        TREF(G,T)       Baseline transfers to households,
        PELLREF(G,T)    Baseline reservation wage,
        PZREF(G,T)      Baseline price of full consumption,
        ZREF_T(G,A)     Baseline post-terminal consumption profile,
        PREF_T(A)       Baseline post-terminal price path,
        PZREF_T(G,A)    Baseline post-terminal full consumption,
        MREF(G)         Baseline present value of consumption;

* We assign demand and income profiles for generation G at time T
* based on endowments and the calibrated consumption profile for
* generation 0. The trick here is to use a GAMS loop over the mapping
* which relates generations (G), ages (A) and time periods (T).

LOOP(MAPG(G,A,T),
        PIREF(G,T)      = PI(A);
        EREF(G,T)       = QREF(G) * OMEGA.L;
        CREF(G,T)       = QREF(G) * CC.L(A);
        ELLREF(G,T)     = QREF(G) * CELL.L(A);
        LREF(G,T)       = QREF(G) * CL.L(A);
        ZREF(G,T)       = QREF(G) * CZ.L(A);
        PELLREF(G,T)    = PREF(G) * ETA.L(A)/LAMDA.L;
        TREF(G,T)       = QREF(G) * PSHR(A)*T0*QREF(A);
        PZREF(G,T)      = PREF(G) * CPZ.L(A);   );

* The last model generation is born in year 150 which means that in
* order to capture the full life cycle of all model generations we
```

```
* need to cover a 50-year "post-terminal" period. We index these
* post-terminal periods by the same index (A) that we use to index
* ages in a life cycle.

* Consumption profiles in post-terminal periods for generation G at
* age A are inferred from the consumption levels in the initial
* period of generations that have the same age.

LOOP((G,A,TLAST)$(YEAR(G)+AGE(A) GT YEAR(TLAST)),
        ZREF_T(G,AA)$(AGE(AA)+YEAR(TLAST) EQ AGE(A)+YEAR(G))
        = QREF(G) * CZ.L(A); );

* Present value prices in post-terminal periods are extrapolated from
* the value of the reference price index in the terminal period.

LOOP((A,TLAST)$AGE(A), PREF_T(A) = PREF(TLAST) / (1 + RBAR_A)**AGE(A); );

LOOP((G,A,TLAST)$(YEAR(G)+AGE(A) GT YEAR(TLAST)),
        PZREF_T(G,AA)$(AGE(AA)+YEAR(TLAST) EQ AGE(A)+YEAR(G))
        = CPZ.L(A)*PREF(G); );

* Present value of consumption by generation, including post-terminal
* consumption by generations who live beyond the model horizon.

MREF(G) = SUM(T,  ZREF(G,T)*PZREF(G,T))
        + SUM(A, PZREF_T(G,A)*ZREF_T(G,A));

*======================================================================
* Distribute assets holdings by type
*======================================================================

SCALARS THETAC          Ratio of capital stock to assets,
        THETAD          Ratio of government deficit to assets,
        THETAB          Ratio of trade deficit to assets;

PARAMETERS
        AOREF(G)        Baseline initial asset holdings,
        ATREF(G)        Baseline terminal asset holdings;

* Value shares of the different asset types as implied by benchmark
* value flows.

THETAC  = (1+RBAR)*K0 / CCMA.L;
THETAB  = (B0*(1+RBAR)/(RBAR-GAMMA)) / CCMA.L;
THETAD  = (-D0*(1+RBAR)/(RBAR-GAMMA)) / CCMA.L;

* Distribute asset types by assuming that all age groups hold the
* different types in same proportion. Use mapping to identify
* generation at time zero from age of reference generation.
```

```
AOREF(G)        = SUM(MAPG(G,A,"0"), CMA.L(A));

* Assets left at end of terminal period for generation G are inferred
* from inital assets.

LOOP((G,GG),    ATREF(G)$(ORD(G) EQ (ORD(GG) + (CARD(G) + 1 - CARD(A))))
                = AOREF(GG) * (1+GAMMA)**CARD(T) / (1+RBAR); );

*======================================================================
* Parameters for counterfactual experiments
*======================================================================

PARAMETERS
        TAXR    Model tax rate on capital earnings,
        TAXL    Model tax rate on labor earnings;

* Initialize model tax rates at benchmark values:

TAXR    = TR0;
TAXL    = TL0;

*======================================================================
* Define share parameters
*======================================================================

PARAMETER       ALPHAC  Goods share of full consumption;

ALPHAC(G,T)$ZREF(G,T) = CREF(G,T)*PREF(T) / (ZREF(G,T)*PZREF(G,T));

*======================================================================
* Model in GAMS/MCP. This model solves for the equilibrium transition
* path subject to terminal conditions that assume the presence of a
* steady state. If there are no exogenous changes the model
* replicates the calibrated consumption profiles. We use this feature
* to check the calibrations and then solve for the results of
* fundamental tax reform
*======================================================================

POSITIVE VARIABLES

* Activity levels. These determine how inputs are converted into
* outputs according to the technology implied by functional forms and
* the benchmark data. The variables here are activity levels and an
* equilibrium requires that each active sector earns zero profit.

        QY(T)           Domestic production,
        QA(T)           Supply of Armington composite,
        QK(T)           Capital stock,
```

```
        QI(T)           Investment,
        QL(G,T)         Labor supply,
        QZ(G,T)         Full consumption,
        QU(G)           Utility,

* Prices. The variables here are the prices that are associated with
* each commodity. An equilibrium requires that prices are such that
* supply equals demand.

        PH(T)           Price of output for domestic use,
        PFX             Price of foreign exchange,
        PY(T)           Composite price of output,
        PA(T)           Price of Armington composite,
        PL(T)           Wage rate (labor in efficiency units),
        PR(T)           Rental rate,
        PK(T)           Price of capital,
        PU(G)           Price of intertemporal utility,
        PKT             Price of post-terminal capital,
        PELL(G,T)       Reservation wage (leisure in units of time),
        PZ(G,T)         Price of full consumption (current value),
        PZT(G,A)        Price of full consumption (current value),

* Incomes. These are agents that receive income from endowments or
* taxes and spend it to maximize utility. The variables here are
* income levels and an equilibrium requires that total income equals
* total expenditure.

        RA(G)           Representative agents by generation,
        GOVT(T)         Government with period-by-period budget,

* Auxiliary variables. These are endogenous variables associated with
* model constraints that relate the transition to the steady state.
* The replacement tax is varying over time to ensure government
* budget balance period by period.

        KT              Terminal capital,
        ZT(G,A)         Post-terminal consumption of goods,
        AT(G)           Terminal bonds,
        TAU(T)          Replacement tax in counterfactual;

EQUATIONS

* Equations asscociated with model variables. These fall into three
* classes. PRF which ensure zero profit in each activity, MKT which
* ensure no excess demand for each commodity, and DEF which ensure
* income balance for each agent.

        PRF_QY(T)       Domestic production,
        PRF_QA(T)       Supply of Armington composite,
```

```
        PRF_QK(T)       Capital stock,
        PRF_QI(T)       Investment,
        PRF_QL(G,T)     Labor supply,
        PRF_QZ(G,T)     Full consumption,
        PRF_QU(G)       Utility,
        MKT_PH(T)       Price of output for domestic use,
        MKT_PFX         Pice of foreign exchange,
        MKT_PA(T)       Price of Armington composite,
        MKT_PL(T)       Wage rate (labor in efficiency units),
        MKT_PR(T)       Rental rate,
        MKT_PK(T)       Price of capital,
        MKT_PU(G)       Price of intertemporal utility,
        MKT_PKT         Price of post-terminal capital,
        MKT_PELL(G,T)   Reservation wage (leisure in units of time),
        MKT_PZ(G,T)     Price of full consumption (current value),
        MKT_PZT(G,A)    Price of full consumption (current value),
        DEF_RA(G)       Representative agents by generation,
        DEF_GOVT(T)     Government - period-by-period balancing,

* Equations associated with auxilliary variables used to impose
* steady-state restrictions on the values of terminal period
* variables, and with simplyfying terms.

        EQ_KT           Terminal capital,
        EQ_ZT(G,A)      Post-terminal consumption of goods,
        EQ_AT(G)        Terminal bonds,
        EQ_TAU(T)       Replacement tax in counterfactual
        EQ_PY(T)        Defines PY to simplify algebra;

*====================================================================
* The following equations define zero profit conditions for each
* activity.
*====================================================================

PRF_QY(T)..
        (PR(T)*(1+TAXR)/(PREF(T)*(1+TR0)))**(R0*(1+TR0)/Y0)*
        (PL(T)*(1+TAXL)/(PREF(T)*(1+TL0)))**(L0*(1+TL0)/Y0)
        =E= PY(T)/PREF(T);

EQ_PY(T)..
        Y0 * (PY(T)/PREF(T))**(1+ETADX)
        =E= X0 * PFX**(1+ETADX)
        + (Y0-X0) * (PH(T)/PREF(T))**(1+ETADX);

PRF_QA(T)..
        (M0 / (C0+I0+G0)) * PFX**(1-SIGMA)
        + ((Y0-X0)/(C0+I0+G0)) * (PH(T)/PREF(T))**(1-SIGMA)
        =E= (PA(T)/PREF(T))**(1-SIGMA);
```

```
PRF_QK(T)..
        PK(T)*K0
        =E= PR(T)*R0 + (PK(T+1)+PKT$TLAST(T))*K0*(1-DELTA);

PRF_QI(T).. PA(T) =E= PK(T+1) + PKT$TLAST(T);

PRF_QL(G,T)$CREF(G,T).. PELL(G,T) =G= PL(T) * PIREF(G,T);

PRF_QZ(G,T)$ZREF(G,T)..
        ALPHAC(G,T) * (PA(T)*(1+TAU(T))/PREF(T))**(1-SIGMA_CL) +
        (1-ALPHAC(G,T)) * (PELL(G,T)/PELLREF(G,T))**(1-SIGMA_CL)
        =E= (PZ(G,T)/PZREF(G,T))**(1-SIGMA_CL);

PRF_QU(G)..
        SUM(T, PZREF(G,T)*ZREF(G,T)*(PZ(G,T)/PZREF(G,T))**(1-1/THETA))
        +SUM(A, PZREF_T(G,A)*ZREF_T(G,A)
        *(PZT(G,A)/PZREF_T(G,A))**(1-1/THETA))
        =E= MREF(G) * PU(G)**(1-1/THETA);

*======================================================================
* The following define market clearance by ensuring no excess demand
* for each commodity
*======================================================================

MKT_PA(T)..
        QA(T) * (Y0-X0+M0) =E= QI(T) * I0
        + SUM(G, CREF(G,T) * QZ(G,T)
        * (PZ(G,T)*PREF(T)/(PA(T)*(1+TAU(T))*PZREF(G,T)))**SIGMA_CL)
        + GOVT(T)/PA(T);

MKT_PELL(G,T)$CREF(G,T)..
        EREF(G,T) =E= QL(G,T) +
        ELLREF(G,T)*QZ(G,T)*(PZ(G,T)*PELLREF(G,T)
        / (PELL(G,T)*PZREF(G,T)))**SIGMA_CL;

MKT_PL(T)..
        SUM(G, QL(G,T) * PIREF(G,T))
        =E= L0 * QY(T) * ((PY(T)*(1+TL0)) / (PL(T)*(1+TAXL)));

MKT_PR(T)..
        QK(T)*R0 =E= R0 * QY(T) * ((PY(T)*(1+TR0)) / (PR(T)*(1+TAXR)));

MKT_PK(T)..
        QK(T-1)*K0*(1-DELTA) + QI(T-1)*I0
        + SUM(G,THETAC*A0REF(G)/(1+RBAR))$TFIRST(T)
        =E= QK(T)*K0;

MKT_PH(T)..
        QY(T) * (PH(T)/PY(T))**ETADX
```

```
        =E= QA(T) * (PA(T)/PH(T))**SIGMA;

MKT_PFX..
        SUM(T, PREF(T) * X0 * QY(T) * (PFX*PREF(T)/PY(T))**ETADX)
        + SUM((G,T), PREF(T)*TREF(G,T))
        + SUM(G, (1-THETAC)*A0REF(G))
        + SUM(T, PREF(T)*QREF(T)*(D0-T0))
        =E= SUM(T, PREF(T) * M0 * QA(T) * (PA(T)/(PREF(T)*PFX))**SIGMA)
        + SUM((G,TLAST), AT(G)*PREF(TLAST)*(1-THETAC)*ATREF(G));

MKT_PKT..
        SUM(TLAST, QK(TLAST)*K0*(1-DELTA) + QI(TLAST)*I0)
        =E= SUM(ATGEN, THETAC * ATREF(ATGEN) * KT);

MKT_PZ(G,T)$ZREF(G,T)..
        QZ(G,T) =E= QU(G) * (PU(G)*PZREF(G,T)/PZ(G,T))**(1/THETA);

MKT_PZT(G,A)$ZREF_T(G,A)..
        ZT(G,A) =E= QU(G) * (PU(G)*PZREF_T(G,A)/PZT(G,A))**(1/THETA);

MKT_PU(G).. QU(G)*PU(G)*MREF(G) =E= RA(G);

*======================================================================
* The following equations define income balance for households and
* government
*======================================================================

DEF_RA(G)..
        RA(G) =E= SUM(T, PELL(G,T) * EREF(G,T))
        + PFX * SUM(T,PREF(T)*TREF(G,T))
        + SUM(TFIRST, PK(TFIRST) * THETAC * A0REF(G)/(1+RBAR))
        + PFX * (1-THETAC)*A0REF(G)
        + SUM(A, PZT(G,A)*ZREF_T(G,A)*ZT(G,A))
        + AT(G) * PFX * (-SUM(TLAST, PREF(TLAST)*(1-THETAC)*ATREF(G)))
        + KT * PKT * (-THETAC*ATREF(G));

DEF_GOVT(T)..
        GOVT(T) =E= PREF(T)*QREF(T)*(D0-T0) * PFX
        + TAXL * PL(T) * L0 * QY(T) * ((PY(T)*(1+TL0)) / (PL(T)*(1+TAXL)))
        + TAXR * PR(T) * R0 * QY(T) * ((PY(T)*(1+TR0)) / (PR(T)*(1+TAXR)))
        + SUM(G, CREF(G,T)* QZ(G,T)
        * (PZ(G,T)*PREF(T)/(PA(T)*(1+TAU(T))*PZREF(G,T)))**SIGMA_CL
        * PA(T) * TAU(T));

*======================================================================
* The following equations describe additional constraints used to
* close the model and select the level of endogenous taxes.
*======================================================================
```

```
* Set the endogenous tax to balance the government budget to balance
* period by period.

EQ_TAU(T).. PA(T) * QREF(T) * GO =E= GOVT(T);

* Select terminal capital stocks so that all generations living past
* the terminal period achieve the same equivalent variation.

EQ_AT(G)$ATGEN(G).. QU(G) - QU(G-1) =E= 0;

* Select the levels of post-terminal consumption of goods and leisure
* so that the present value price declines with the steady-state
* interest rate.

EQ_ZT(G,A)$ZREF_T(G,A)..
        SUM(TLAST, PZ(G-(ORD(A)-1),TLAST))
        =E= PZT(G,A) * (1+RBAR)**(ORD(A)-1);

* Scale the level of the terminal capital staock to achieve
* steady-state growth in last period investment.

EQ_KT.. SUM(TLAST(T), QI(T)/QI(T-1)) =E= 1 + GAMMA;

*======================================================================
* Assign initial values and bounds for activity levels, prices, and
* auxiliary variables:
*======================================================================

QY.L(T)         = QREF(T);
QA.L(T)         = QREF(T);
QK.L(T)         = QREF(T);
QI.L(T)         = QREF(T);
QL.L(G,T)       = LREF(G,T);
PY.L(T)         = PREF(T);
PH.L(T)         = PREF(T);
PFX.L           = 1;
PA.L(T)         = PREF(T);
PL.L(T)         = PREF(T);
PZ.L(G,T)       = PZREF(G,T);
PZT.L(G,A)      = PZREF_T(G,A);
PR.L(T)         = PREF(T);
PK.L(T)         = PREF(T)*(1+RBAR);
PELL.L(G,T)     = PELLREF(G,T);
PZT.L(G,A)      = PZREF_T(G,A);
LOOP(TLAST, PKT.L= PK.L(TLAST) / (1+RBAR));
KT.L            = 1;
KT.LO           = -INF;
AT.L(G)         = 1;
AT.LO(G)        = -INF;
```

```
ZT.L(G,A)       = 1$ZREF_T(G,A);
TAU.LO(T)       = -INF;
TAU.L(T)        = 0;
QU.L(G)         = 1;
PU.L(G)         = 1;
QZ.L(G,T)       = 1$ZREF(G,T);
GOVT.L(T)       = PREF(T) * QREF(T) * G0;
RA.L(G)         = MREF(G);

* Numeraire:

PA.FX(TFIRST)   = 1;

MODEL OLG /     PRF_QY.QY,PRF_QA.QA,PRF_QK.QK,PRF_QI.QI,
                PRF_QL.QL,PRF_QZ.QZ,PRF_QU.QU,MKT_PH.PH,
                MKT_PFX.PFX,MKT_PA.PA,MKT_PL.PL,MKT_PR.PR,
                MKT_PK.PK,MKT_PU.PU,MKT_PKT.PKT, MKT_PELL.PELL,
                MKT_PZ.PZ,MKT_PZT.PZT,DEF_RA.RA,DEF_GOVT.GOVT,
                EQ_KT.KT,EQ_ZT.ZT,EQ_AT.AT,EQ_TAU.TAU,EQ_PY.PY /;

*======================================================================
* Replicate the benchmark equilibrium:
*======================================================================

OLG.ITERLIM=0;
SOLVE OLG USING MCP;

*======================================================================
* Run counterfactual: reduce taxes on capital and labor income:
*======================================================================

* Parameters for reporting results from counterfactual experiment:

PARAMETERS
        WCHANGE Welfare change (% equivalent variation by year of birth),
        WGAIN   Base year total welfare gain (1996 USD billion),
        TDEF    Trade deficit (% of baseline level),
        KSTOCK  Capital stock (% change from baseline level),
        LSUPPLY Aggregate labor supply (% change from baseline level),
        CONSTAX Consumption tax rate (%);

* Reduce tax on capital income or tax on labor income by $100 billion:

SETS    SCENARIO                /CAPITAL,LABOR/,
        CAPTAX(SCENARIO)        /CAPITAL/
        LABTAX(SCENARIO)        /LABOR/;

LOOP(SCENARIO,
```

```
        TAU.L(T)        = 0;
        TAXR            = TR0;
        TAXL            = TL0;

        IF (CAPTAX(SCENARIO), TAXR = TR0 - 100*1E-3/R0; );
        IF (LABTAX(SCENARIO), TAXL = TL0 - 100*1E-3/L0; );

* Solve the model

        OLG.ITERLIM=10000;
        SOLVE OLG USING MCP;

*=======================================================================
* Report results from tax reform
*=======================================================================

        WCHANGE(G,SCENARIO)     = 100 * (QU.L(G) - 1);
        WGAIN(SCENARIO)
        = SUM(G$(ORD(G) EQ CARD(G)),(QU.L(G)-1))
        * SUM(A, (CC.L(A)+(ETA.L(A)/LAMDA.L)*CELL.L(A)/
        PREF(A))/QREF(A)) / 1E-3;

        TDEF(T,SCENARIO)
        = 100 * (( M0 * QA.L(T) * (PA.L(T)/(PREF(T)*PFX.L))**SIGMA
        - X0 * QY.L(T) * (PFX.L*PREF(T)/PY.L(T))**ETADX)
        / (QREF(T)*B0) -1);
        KSTOCK(T,SCENARIO)      = 100 * (QK.L(T) / QREF(T) - 1);
        LSUPPLY(T,SCENARIO)
        = 100 * (L0 * QY.L(T) * ((PY.L(T)*(1+TL0)) / (PL.L(T)*(1+TAXL)))
        / (L0*QREF(T)) - 1);
        CONSTAX(T,SCENARIO)     = 100 * TAU.L(T);

* End scenario loop
);

DISPLAY PSHR,QREF,PREF,TR0,TL0,K0,PIREF,EREF,CREF,ELLREF,LREF,ZREF,
        TREF,PELLREF,PZREF,ZREF_T,PREF_T,PZREF_T,MREF,RHO.L,RHO_A,
        OMEGA.L,PELLRATIO,THETAC,THETAD,THETAB,WCHANGE,WGAIN,TDEF,
        KSTOCK,LSUPPLY,CONSTAX;
```