## Geometric Sums and Terminal Approximation of the Ramsey Model

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If we assume a constant consumption growth rate of  $\gamma$  from periods T to  $\infty$ , the Cobb-Douglas utility function can be written as:

$$U = \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho}\right)^{t} \log(C_{t})$$
  
=  $\sum_{t=0}^{T-1} \left(\frac{1}{1+\rho}\right)^{t} \log(C_{t}) + \sum_{t=T}^{\infty} \left(\frac{1}{1+\rho}\right)^{t} \log\left(C_{T}(1+\gamma)^{t-T}\right)$   
=  $\sum_{t=0}^{T-1} \left(\frac{1}{1+\rho}\right)^{t} \log(C_{t}) + \left(\frac{1}{1+\rho}\right)^{T} \left[\log(C_{T}) \sum_{\tau=0}^{\infty} \left(\frac{1}{1+\rho}\right)^{\tau} + \log(1+\gamma) \sum_{\tau=0}^{\infty} \tau \left(\frac{1}{1+\rho}\right)^{\tau}\right]$   
=  $\sum_{t=0}^{T-1} \left(\frac{1}{1+\rho}\right)^{t} \log(C_{t}) + \left(\frac{1}{1+\rho}\right)^{T} \left[\log(C_{T}) \frac{1+\rho}{\rho} + \log(1+\gamma) \sum_{\tau=0}^{\infty} \tau \left(\frac{1}{1+\rho}\right)^{\tau}\right]$ 

Where then can write:

$$U = \sum_{t=0}^{T} \beta_t \log(C_t) + \kappa$$

where

$$\beta_t = \begin{cases} \left(\frac{1}{1+\rho}\right)^t & t < T\\ \left(\frac{1}{1+\rho}\right)^T \frac{1}{\rho} & t = T \end{cases}$$

and

$$\kappa = \left(\frac{1}{1+\rho}\right)^T \sum_{\tau=0}^{\infty} \tau \left(\frac{1}{1+\rho}\right)^{\tau}$$

I prefer to take the steady-state growth and interest rates as model inputs in place of the discount factor. If the assumed steady-state growth rate is  $\gamma$  and the steady-state interest rate is r, you have a discount rate given by:

$$\rho = \frac{1+r}{1+\gamma} - 1 = \frac{r-\gamma}{1+r}$$

and

$$\beta_t = \begin{cases} \left(\frac{1+\gamma}{1+r}\right)^t & t < T\\ \left(\frac{1+\gamma}{1+r}\right)^T \frac{1+r}{r-\gamma} & t = T \end{cases}$$

If you are working with a CES utility function with an intertemporal elasticity equal to  $1/\theta$ , the algebra is a bit different. Using the calibrated share form and imposing the terminal assumption that  $C_t = (1+g)^{t-T}C_T \quad \forall t \geq T$ , we have:

$$\begin{aligned} U &= \left[ \sum_{t=0}^{\infty} \left( \frac{1+\gamma}{1+r} \right)^t \left( \frac{C_t}{(1+\gamma)^t} \right)^{1-\theta} \right]^{1/1-\theta} \\ &= \left[ \sum_{t=0}^{T-1} \left( \frac{1+\gamma}{1+r} \right)^t \left( \frac{C_t}{(1+\gamma)^t} \right)^{1-\theta} + \left( \frac{1+\gamma}{1+r} \right)^T \sum_{\tau=0}^{\infty} \left( \frac{1+\gamma}{1+r} \right)^\tau \left( \frac{C_T(1+\gamma)^\tau}{(1+\gamma)^\tau} \right)^{1-\theta} \right]^{1/1-\theta} \\ &= \left[ \sum_{t=0}^{T-1} \left( \frac{1+\gamma}{1+r} \right)^t \left( \frac{C_t}{(1+\gamma)^t} \right)^{1-\theta} + \left( \frac{1+\gamma}{1+r} \right)^T C_T^{1-\theta} \sum_{\tau=0}^{\infty} \left( \frac{1+\gamma}{1+r} \right)^\tau \right]^{1/1-\theta} \\ &= \left[ \sum_{t=0}^{T-1} \left( \frac{1+\gamma}{1+r} \right)^t \left( \frac{C_t}{(1+\gamma)^t} \right)^{1-\theta} + \frac{1+r}{r-\gamma} \left( \frac{1+\gamma}{1+r} \right)^T C_T^{1-\theta} \right]^{1/1-\theta} \\ &= \left( \sum_{t=0}^{T} \beta_t C_t^{1-\theta} \right)^{1/1-\theta} \end{aligned}$$

where

$$\beta_t = \begin{cases} \left(\frac{(1+\gamma)^{\theta}}{1+r}\right)^t & t < T\\ \frac{1+r}{r-\gamma} \left(\frac{(1+\gamma)^{\theta}}{1+r}\right)^T & t = T \end{cases}$$

Note that this expression corresponds precisely to the Cobb-Douglas result when we have  $\theta = 1$ .